EXPLANATIONS FROM CALCULUS I

Calculus can be used to explain many important facts in mathematics. Before calculus, these facts are simply asserted without justification. Now we can see why they are true.

The area of a circle. We can approximate the area of a circle of radius r by inscribing a regular polygon inside it. A regular n-gon can be divided into n wedges, each of which is an isosceles triangle with angle $\theta = \frac{2\pi}{n}$. The area of one triangle is $A_T = \frac{1}{2}r^2 \sin\theta$. The area of the entire polygon is $A_P = n\frac{1}{2}r^2 \sin\theta$. We take a limit as $n \to \infty$, or equivalently, as $\theta \to 0$. Then the area of the circle is

$$A_C = \lim_{n\to\infty} n\frac{1}{2}r^2 \sin\theta = \lim_{\theta\to 0} \frac{2\pi}{\theta}\frac{1}{2}r^2 \sin\theta = \pi r^2 \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \pi r^2.$$

The perimeter of a circle. Consider increasing the radius r of a circle by a small amount Δr. Then the area will increase by a small amount ΔA. The area added forms a thin shell around the circle. The area of this shell will be approximately equal to the circumference of the circle $C(r)$ times Δr. The differential approximation $\Delta A \approx dA$ yields $\Delta A \approx C(r)\,\Delta r = dA = \frac{dA}{dr}dr$. Since $\Delta r = dr$, we find $C(r) = \frac{dA}{dr} = 2\pi r$. Thus the circumference is the derivative of the area of a circle. Essentially the same argument shows that the surface area of a sphere is the derivative of its volume, $S(r) = \frac{dV}{dr} = \frac{d}{dr}\frac{4}{3}\pi r^3 = 4\pi r^2$.

The vertex of a parabola. The graph of $p(x) = ax^2 + bx + c$ is a parabola. The vertex is an extreme point. Thus the derivative there should be zero. We find $p'(x) = 2ax + b = 0$ implies $x = -\frac{b}{2a}$. This is the x-coordinate of the vertex of the parabola. It is possible to derive this without calculus, but this derivation is simple and elegant.

Rectangle with maximum area. Given a fixed perimeter, what rectangle encloses the largest area? It may seem obvious that the answer is a square, but how can we know for sure? Let the perimeter be P and the lengths of one pair of sides be x. Then the lengths of the other pair of sides must be $\frac{P-2x}{2} = \frac{P}{2} - x$. The area of the rectangle is $A = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$. Then $A' = \frac{P}{2} - 2x = 0$, so $x = \frac{P}{4}$. Then all sides have the same length, so the rectangle is a square.

Motion in one dimension. Near the surface of the Earth, gravitational acceleration is approximately constant. Constant acceleration yields the differential equation $\frac{d^2s}{dt} = a$. Finding the antiderivative, we have $\frac{ds}{dt} = at + v_0$ with v_0 being the (constant) initial velocity. Finding the antiderivative again, we have $s(t) = \frac{1}{2}at^2 + v_0 t + s_0$, with s_0 the initial position. If we have downward gravitational acceleration $a = -g$ in the vertical (y) direction, this becomes $y(t) = -\frac{1}{2}gt^2 + v_{y0}t + y_0$. This formula is presented in basic physics, but the justification requires calculus.

Motion in two dimensions. What if an object moves in more than one dimension? We can consider its motion in each component separately. The same laws apply in both cases. If there is no acceleration in the horizontal direction (as is essentially true with gravity), the velocity will be constant. The position will be $x(t) = v_{x0}t + x_0$, with initial velocity v_{x0} and initial position x_0. We can combine these two functions into one vector-valued function for an object's position $\overrightarrow{s}(t) = (x, y) = \left(v_{x0}t + x_0, -\frac{1}{2}gt^2 + v_{y0}t + y_0\right)$. The two component equations can be combined into one by eliminating the parameter t. Solving $x = v_{x0}t + x_0$, for t, we find $t = \frac{x - x_0}{v_{x0}}$. Substituting into the other equation, we find $y = -\frac{1}{2}g\left(\frac{x-x_0}{v_{x0}}\right)^2 + v_{y0}\left(\frac{x-x_0}{v_{x0}}\right) + y_0$. This is a quadratic equation, so an object launched near Earth's surface with constant velocity will travel along a parabola.

Polynomials of odd degree have a real zero. Any polynomial $p(x)$ is continuous at every point of its domain. If the leading coefficient of $p(x)$ is positive, then $\lim_{x\to\infty} p(x) = \infty$ and $\lim_{x\to-\infty} p(x) = -\infty$. If the leading coefficient is negative, the signs reverse. Either way, sign changes at some point. By the Intermediate Value Theorem, $p(x)$ must have a zero. This is an important piece of the proof of the Fundamental Theorem of Algebra, which says that every non-constant polynomial with complex coefficients has a complex zero.