EXPLANATIONS FROM CALCULUS II

Calculus can be used to explain many important facts in mathematics. Before calculus, these facts are simply asserted without justification. Now we can see why they are true.

Integrals for formulas. Many familiar formulas for area and volume can be computed with an appropriate definite integral. The integral is set up by solving the appropriate equation and using the appropriate integration technique. The table below lists a number of examples.

Formula For	Integral	Method Used	Formula
Perimeter of a Circle	$P = 4\int_0^r \sqrt{1+\left(\frac{d}{dx}\sqrt{r^2-x^2}\right)^2}dx$	arclength	$P = 2\pi r$
$(x, y) = (r\cos\theta, r\sin\theta)$	$P = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}d\theta$	arclength	$P = 2\pi r$
Area of a Circle $x^2+y^2=r^2$	$A = 4\int_0^r \sqrt{r^2-x^2}dx$	trig sub.	$A = \pi r^2$
Ellipse Area $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$	$A = 4\int_0^a \sqrt{b^2\left(1-\left(\frac{x}{a}\right)^2\right)}dx$	trig sub.	$A = \pi ab$
Volume of a Cone	$V = \int_0^h \pi\left(\frac{r}{h}x\right)^2 dx$	disk method	$V = \frac{1}{3}\pi r^2 h$
radius r, height h	$V = \int_0^r 2\pi x\left(h-\frac{h}{r}x\right)dx$	shell method	$V = \frac{1}{3}\pi r^2 h$
Volume of a Sphere	$V = \int_{-r}^r \pi\left(\sqrt{r^2-x^2}\right)^2 dx$	disk method	$V = \frac{4}{3}\pi r^3$
rotate $x^2+y^2=r^2$ about $y=0$	$V = \int_0^r 2\pi x\left(2\sqrt{r^2-x^2}\right)dx$	shell method	$V = \frac{4}{3}\pi r^3$

Differential equations for functions. Differential equations can be used to explain (or even define) several basic functions.

Type of Function	Differential Equation	Technique Used	Solution
Exponential	$\frac{dy}{dx} = ky$	Separable, Homogeneous DE	$y = Ae^{kx}$
Logistic	$\frac{dy}{dx} = ky(1-y)$	Separable, Homogeneous DE	$y = \frac{1}{Ae^{-kx}+1}$

Growth rates. Given two functions f and g that go to infinity as $n \to \infty$, we can determine which grows faster by computing $L = \lim_{n\to\infty}\frac{f(n)}{g(n)}$. If $L = \infty$, f grows faster than g. This can be used to show that

$$\ln\ln n \ll \ln n \ll n^p,\ p \in (0,1) \ll n \ll n\ln n \ll n^p,\ p > 1 \ll e^n \ll n! \ll n^n \ll 2^{2^n}$$

Decimal to fraction. Every fraction has a decimal expansion that either terminates or repeats. Given a repeating decimal, we can represent it as a geometric series and find its sum. Let $s = .\overline{a_1a_2...a_n}$. Then $10^n s = a_1a_2...a_n.\overline{a_1a_2...a_n}$. Subtracting, we find $10^n s - s = a_1a_2...a_n$. Hence $s = \frac{a_1a_2...a_n}{10^n-1} = \frac{a_1a_2...a_n}{99...9}$. This is the fractional form of the repeating decimal.

Series for irrational numbers. Using Taylor series and integrals of Taylor series, we can determine series that converge to many famous irrational numbers. These can be used to find arbitrarily close decimal approximations for these numbers.

Power Series	Plug in	Resulting Series
$e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$	$x = 1$	$e = \sum_{n=0}^\infty \frac{1}{n!} = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+...$
$\arctan x = \sum_{n=0}^\infty (-1)^n\frac{x^{2n+1}}{2n+1}$	$x = 1$	$\frac{\pi}{4} = \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+...$
	$x = \frac{1}{\sqrt{3}}$	$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)3^n} = \frac{1}{\sqrt{3}}\left(1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 9}-\frac{1}{7\cdot 27}+...\right)$
$\ln(1+x) = \sum_{n=0}^\infty (-1)^n\frac{x^n}{n}$	$x = 1$	$\ln 2 = \sum_{n=0}^\infty \frac{(-1)^n}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+...$
$\frac{1}{2}\ln\left\vert\frac{x+1}{x-1}\right\vert = \sum_{n=0}^\infty \frac{x^{2n+1}}{2n+1}$	$x = \frac{1}{3}$	$\ln 2 = \frac{2}{3}\sum_{n=0}^\infty \frac{1}{(2n+1)9^n} = \frac{2}{3}\left(1+\frac{1}{3\cdot 9}+\frac{1}{5\cdot 81}+\frac{1}{7\cdot 729}+...\right)$
$(1+x)^m = \sum_{n=0}^\infty \binom{m}{n}x^n$	$m = \frac{1}{2}$	
$\sqrt{1+x} = \sum_{n=0}^\infty \binom{1/2}{n}x^n$	$x = 1$	$\sqrt{2} = \sum_{n=0}^\infty \frac{(2k-3)!!}{(2k)!!} = 1+\frac{1}{2}-\frac{1}{8}+\frac{3}{48}-\frac{15}{384}+\frac{105}{3840}-...$