DERIVING AN EXPLICIT FORMULA FOR THE FIBONACCI NUMBERS

Define a power series whose coefficients are the Fibonacci numbers. This is called a generating function. Further, consider multiplying it by x and x^2.

$$\begin{aligned} F(x) &= x + x^2 + 2x^3 + 3x^4 + 5x^5 + ... \\ x \cdot F(x) &= \quad\;\; x^2 + \;\; x^3 + 2x^4 + 3x^5 + ... \\ x^2 \cdot F(x) &= \qquad\qquad\;\; x^3 + \;\; x^4 + 2x^5 + ... \end{aligned}$$

The first line is almost the sum of the latter two. In particular,

$$F(x) = x \cdot F(x) + x^2 \cdot F(x) + x$$

$$F(x) = \frac{x}{1 - x - x^2}$$

This is the closed form of the generating function. Factor the denominator by finding where it is zero. We find

$$1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, since $\alpha\beta = -1$ and $\alpha + \beta = 1$. The generating function can be decomposed into partial fractions.

$$F(x) = \frac{x}{1 - x - x^2} = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x}$$

Solving for the coefficients, we find

$$x = A(1 - \beta x) + B(1 - \alpha x)$$

$$= (A + B) + (-A\beta - B\alpha)x.$$

Thus $A+B=0$, and $-A\left(\frac{1-\sqrt{5}}{2}\right)-B\left(\frac{1+\sqrt{5}}{2}\right)=1$, so $A=\frac{1}{\sqrt{5}}$ and $B=-\frac{1}{\sqrt{5}}$. But the partial fractions can be expressed as geometric series, as $\frac{1}{1-\alpha x}=\sum(\alpha x)^n$. Thus

$$F(x)=\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}(\alpha x)^n-\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}(\beta x)^n$$

The coefficient of x^n in $F(x)$ is

$$f_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

This is an explicit formula for all Fibonacci numbers!

Note further that $\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)<\frac{1}{2}$ and $\left(\frac{1-\sqrt{5}}{2}\right)^n\to 0$, so

$$f_n\approx\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

Specifically, the n^{th} Fibonacci number is the integer closest to the quantity on this right. This also implies that the ratio of consecutive Fibonacci numbers

$$\frac{f_{n+1}}{f_n}\to\frac{1+\sqrt{5}}{2}$$

This quantity is called the golden ratio. It is widely used in art and architecture, as it is believed to be aesthetically pleasing.