EVALUATING THE SUM OF INVERSE SQUARES

The integral test tells us that $\sum \frac{1}{n^2}$ converges. Summing up many terms provides an approximation of $\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.64493...$, and the integral test provides bounds on the error. But what is the exact value?

Consider the Taylor series for sine. Dividing by x, we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ...$$

Now in any finite polynomial, a root of $x = r$ corresponds to a factor of $(x - r)$ in the polynomial. By the Fundamental Theorem of Algebra, any finite polynomial can be written as a product of such (complex) factors, along with some constant multiple.

We assume that the same holds for infinite polynomials. Now $\frac{\sin x}{x}$ has all the roots of $\sin x$ except 0, namely $x = n \cdot \pi$, where $n = \pm 1, \pm 2, \pm 3, ...$ We express the factors in a slightly different form, $1 - \frac{x}{r}$, noting that $x = r$ still produces a zero.

$$\frac{\sin x}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\cdots$$

Multiplying together the pairs of factors for the roots $x = \pm r$, we find

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots$$

Now we have two different polynomial expressions for $\frac{\sin x}{x}$, which should both have the same coefficients. We can find the coefficients by multiplying out the polynomial. The constant term must be 1, and since $\lim_{x \to 0} \frac{\sin x}{x} = 1$, we do not need an additional constant multiple in our polynomial. Each x^2 term is formed from an x^2 term of one of the factors and 1's from all the other factors. Thus we have x^2 term

$$-\frac{x^2}{\pi^2} - \frac{x^2}{4\pi^2} - \frac{x^2}{9\pi^2} - ... = -\frac{1}{\pi^2}\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)x^2.$$

But the x^2 term of the earlier infinite series for $\frac{\sin x}{x}$ was $-\frac{x^2}{3!} = -\frac{1}{6}x^2$. Equating the coefficients, we find

$$-\frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{6}.$$

Solving, we see

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

A similar argument can be used to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$