STIRLING'S APPROXIMATION FOR $n!$

Some computer algorithms have complexity proportional to $n!$, so it is important to know how quickly the factorial function grows. We can compare the growth rates of functions by computing the limit of the ratio of the functions. For example, by repeatedly applying L'Hopital's rule to $\lim_{n\to\infty} \frac{e^n}{n^p}$, we find that e^n grows faster than any polynomial. However, to use L'Hopital's rule we need the functions to be differentiable, which $n!$ is not. Hence it would be useful to have an approximation for $n!$ that is differentiable. Stirling's approximation gives exactly that.

To approximate $n!$, we consider the logarithm of $n!$.

$$\ln(n!) = \ln 1 + \ln 2 + ... + \ln(n-1) + \ln n$$

This resembles a Riemann sum for $\int_1^n \ln x \cdot dx$ with $\Delta x = 1$. A better approximation for this integral can be found using the Trapezoid Rule, which gives a slight underestimate.

$$T = \frac{1}{2}\ln 1 + \ln 2 + ... + \ln(n-1) + \frac{1}{2}\ln n$$

The integral can also be evaluated directly using integration by parts as follows.

$$u = \ln x \quad dv = dx$$

$$du = \frac{1}{x}dx \quad v = x$$

$$\int \ln x \cdot dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - x + C$$

$$\int_1^n \ln x \cdot dx = n\ln n - n + 1$$

Hence

$$\ln(n!) = T + \frac{1}{2}\ln n \approx n\ln n - n + 1 + \frac{1}{2}\ln n$$

Writing the right side as a logarithm, we find

$$\ln(n!) \approx \ln n^n - \ln e^n + \ln e + \ln\sqrt{n} = \ln e\sqrt{n}\left(\frac{n}{e}\right)^n$$

Exponentiating, we find

$$n! \approx e\sqrt{n}\left(\frac{n}{e}\right)^n$$

A more careful analysis of the error leads to improving the constant term from $e = 2.718...$ to $\sqrt{2\pi} = 2.507....$ Thus Stirling's approximation for $n!$ is

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

In particular,

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

(and the limit converges quickly). Using Stirling's approximation, it is possible to show that $e^n < n! < n^n = e^{n\ln n}$ for n sufficiently large ($n \geq 6$ works).