GENERALIZING ALGEBRA AND CALCULUS WITH VECTORS

The main goal of vector calculus is to generalize the concepts of algebra and calculus to larger numbers of dimensions. The following table has a list of algebraic concepts and their generalizations. Many of the generalizations exist in n dimensions, but for simplicity we restrict their statement to three dimensions.

Algebra Concept	Vector Generalization ($n = 3$)
point (x, y)	point/vector $\overrightarrow{x} = (x, y, z)$
distance to the origin $D = \sqrt{x^2 + y^2}$	magnitude of a vector $\lvert\overrightarrow{x}\rvert = \sqrt{x^2 + y^2 + z^2}$
distance between two points $\overline{XY} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	distance between two points $\left\lvert\overrightarrow{a} - \overrightarrow{b}\right\rvert = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
conic sections (four below)	quadric surfaces
circle $(x - a)^2 + (y - b)^2 = r^2$	sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$
ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
parabola $y = x^2$ or $x = y^2$	paraboloid $z = x^2 + y^2$
hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	hyperboloid (one or two sheets)
	other quadric surfaces: elliptic cone, elliptic paraboloid, hyperbolic paraboloid
polar coordinates (r, θ) $(x, y) = (r\cos\theta, r\sin\theta)$ $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$	cylindrical coordinates (r, θ, z) $(x, y, z) = (r\cos\theta, r\sin\theta, z)$
	spherical coordinates (ρ, θ, ϕ) $(x, y, z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$
cosine of an angle	$\cos\theta = \frac{\overrightarrow{x} \cdot \overrightarrow{y}}{\lvert\overrightarrow{x}\rvert\lvert\overrightarrow{y}\rvert}$
triangle inequality $\overline{AC} \leq \overline{AB} + \overline{BC}$	$\lvert\overrightarrow{x} + \overrightarrow{y}\rvert \leq \lvert\overrightarrow{x}\rvert + \lvert\overrightarrow{y}\rvert$
work $W = Fd$	work $W = \overrightarrow{F} \cdot \overrightarrow{d}$
line (point-slope) $y - y_0 = m(x - x_0)$	line (parametrized) $\overrightarrow{x} = \overrightarrow{m}t + \overrightarrow{x_0}$
plane	plane (parametrized) $\overrightarrow{x} = \overrightarrow{a}s + \overrightarrow{b}t + \overrightarrow{x_0}$
	plane (normal form) $\overrightarrow{n} \cdot (\overrightarrow{x} - \overrightarrow{x_0}) = 0$
parallelogram	parallelogram $\overrightarrow{x} = \overrightarrow{a}s + \overrightarrow{b}t + \overrightarrow{x_0}$, $s, t \in [0, 1]$
	parallelepiped $\overrightarrow{x} = \overrightarrow{a}s + \overrightarrow{b}t + \overrightarrow{c}u + \overrightarrow{x_0}$
open interval $\{x \mid \lvert x - a\rvert < r\}$	open ball $B_r(\overrightarrow{a}) = \{\overrightarrow{x} \mid \lvert\overrightarrow{x} - \overrightarrow{a}\rvert < r\}$
	open set S: $\forall\overrightarrow{x} \in S$, $\exists\delta > 0$ s.t. $B_\delta(\overrightarrow{x}) \subseteq S$

Calculus concepts dealing with optimization and approximation of functions generalize to vector calculus as follows.

Optimization/Approximation Concept	Vector Calculus Concept
quadratic function $f(x) = ax^2$	quadratic form $q(\overrightarrow{x}) = \overrightarrow{x}^T A \overrightarrow{x}$
quadratic opens up: $f(x) = ax^2$, $a > 0$	QF positive definite: $q(\overrightarrow{x}) > 0$ for $\overrightarrow{x} \neq 0$
quadratic opens down: $f(x) = ax^2$, $a < 0$	QF negative definite: $q(\overrightarrow{x}) < 0$ for $\overrightarrow{x} \neq 0$
	QF indefinite: $\exists x_1, x_2$ s.t. $q(\overrightarrow{x}_1) < 0 < q(\overrightarrow{x}_2)$
second derivative $\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$	2nd-order partial derivative $\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$
	Hessian $Hf(\overrightarrow{a}) = \left[\frac{\partial^2 f}{\partial x_j \partial x_i}\right](\overrightarrow{a}) = J\overrightarrow{\nabla} f(\overrightarrow{x})$
tangent line $y = f(a) + f'(a)(x - a)$	tangent plane $\overrightarrow{\nabla} f(\overrightarrow{x}) \cdot (\overrightarrow{x} - \overrightarrow{a}) = 0$
linearization $L(x) = f(a) + f'(a)(x - a)$	linear approximation $L(\overrightarrow{x}) = f(\overrightarrow{a}) + Jf(\overrightarrow{a})(\overrightarrow{x} - \overrightarrow{a})$
	first-degree Taylor polynomial $p_1(\overrightarrow{x}) = f(\overrightarrow{a}) + \overrightarrow{\nabla} f(\overrightarrow{a}) \cdot (\overrightarrow{x} - \overrightarrow{a})$
Taylor polynomial (degree n) $p(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) \cdot x^k$	2nd-degree Taylor polynomial (let $\overrightarrow{h} = \overrightarrow{x} - \overrightarrow{a}$) $p_2(\overrightarrow{x}) = f(\overrightarrow{a}) + \overrightarrow{\nabla} f(\overrightarrow{a}) \cdot \overrightarrow{h} + \frac{1}{2}\overrightarrow{h}^T Hf(\overrightarrow{a})\overrightarrow{h}$
local minimum $\exists$ interval I about a, $f(x) \geq f(a)\ \forall x \in B$	local minimum $\exists$ open ball B about $\overrightarrow{a}$, $f(\overrightarrow{x}) \geq f(\overrightarrow{a})\ \forall \overrightarrow{x} \in B$
local maximum $\exists$ interval I about a, $f(x) \leq f(a)\ \forall x \in B$	local maximum $\exists$ open ball B about $\overrightarrow{a}$, $f(\overrightarrow{x}) \leq f(\overrightarrow{a})\ \forall \overrightarrow{x} \in B$
Extreme Value Theorem A continuous function on a closed, bounded interval achieves a min and max value on it.	Extreme Value Theorem A continuous function on a closed, bounded set achieves a min and max value on it.
First Derivative Test If f has a local extremum at a, then $f'(a) = 0$ or is undefined.	First Derivative Test If f has a local extremum at $\overrightarrow{a}$, then $\overrightarrow{\nabla} f(\overrightarrow{a}) = \overrightarrow{0}$ or is undefined.
Second Derivative Test: Given $f'(a) = 0$, If $f''(a) > 0$, a is a local minimum. If $f''(a) < 0$, a is a local maximum.	Second Derivative Test: Given $\overrightarrow{\nabla} f(\overrightarrow{a}) = \overrightarrow{0}$, If $Hf(\overrightarrow{a})$ is pos. def., $\overrightarrow{a}$ is a local minimum. If $Hf(\overrightarrow{a})$ is neg. def., $\overrightarrow{a}$ is a local maximum. If $Hf(\overrightarrow{a})$ is indefinite, $\overrightarrow{a}$ is a saddle point.

The various forms of integration studied in calculus I and II have natural generalizations in vector calculus.

Integral Calculus Concept	Vector Calculus Generalization
arclength $L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2} dt$	arclength $L = \int_a^b \left\| \overrightarrow{f}'(t) \right\| dt$
Riemann integral $\int_a^b f(x)\, dx$	line integral $\int_C u \cdot dL = \int_a^b u\left(\overrightarrow{f}(t)\right) \left\| \overrightarrow{f}'(t) \right\| dt$
	line integral $\int_C \overrightarrow{F} \cdot d\overrightarrow{x} = \int_a^b \overrightarrow{F}\left(\overrightarrow{f}(t)\right) \cdot \overrightarrow{f}'(t)\, dt$
volume by slicing (e.g. disk, washer)	double integral $\iint_R f \cdot dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dydx$
$V = \int_a^b A(x)\, dx$	triple int. $\iiint_S f \cdot dA = \iint_R \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\, dz \right) dA$
various integral properties	analogous integral properties
u-substitution $\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$	change of variables for double integrals $\iint_R g(x, y) \cdot dA = \iint_{R^*} g\left(\overrightarrow{f}(s, t)\right) \left\| \frac{\partial \overrightarrow{f}(s,t)}{\partial(s,t)} \right\| dsdt$
shell method $V = \int 2\pi x \cdot f(x)\, dx$	polar coordinate integral $\iint_R g(x, y)\, dx \cdot dy$ $= \int_{\theta_1}^{\theta_2} \int_{f_1(\theta)}^{f_2(\theta)} g(r\cos\theta, r\sin\theta)\, r \cdot drd\theta$
Fundamental Theorem of Calculus $\int_a^b f'(x)\, dx = f(b) - f(a)$	Fundamental Theorem of Line Integrals $\int_a^b \overrightarrow{\nabla} f(\overrightarrow{x}) \cdot d\overrightarrow{x} = f\left(\overrightarrow{b}\right) - f(\overrightarrow{a})$
	Green's Theorem $\oint_{\partial R} (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$
	Stokes' Theorem $\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot n\, dA$
	Divergence Theorem $\iiint_D \nabla \cdot F\, dV = \iint_S F \cdot n\, dS$