ARCLENGTH AND INTEGRALS

To find the length of a straight line, you can use the distance formula, which comes from the Pythagorean Theorem. But how can we find the length of a curve, assuming this concept can be defined appropriately? As usual in calculus, if we don't know how to define something exactly, we use an approximation instead.

PARAMETRIC CURVES Subdivide a parametric curve $(x, y) = (f(t), g(t))$, $a \leq t \leq b$, and approximate each piece with a short line segment. Let ds be the length of a typical segment. We can relate it to the horizontal and vertical lengths dx and dy using the Pythagorean Theorem.

$$ds^2 = dx^2 + dy^2$$

We divide by dt^2, solve for ds, and integrate to find the length L of the curve.

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$L = \int ds = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}dt.$$

We make this the definition of the length of a plane curve. We require that $f(t)$ and $g(t)$ are continuously differentiable functions and that f' and g' are not simultaneously 0. Thus the curve is smooth, so that the parametrization doesn't backtrack on itself.

Calculating arclength leads to many interesting integrals which illustrate many different integration techniques. Here are some examples.

Example. A circle with radius r can be parametrized by $(x, y) = (r\cos t, r\sin t)$, $0 \leq t \leq 2\pi$. Its arclength is

$$L = \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}dt = \int_0^{2\pi} \sqrt{r^2\left(\sin^2 t + \cos^2 t\right)}dt = \int_0^{2\pi} r \cdot dt = 2\pi r$$

Thus we obtain the expected formula for the circumference of a circle. This provides a justification for a formula that has previously only been taken for granted.

Example. Find the length of the curve $(x, y) = \left(t^3, \frac{3}{2}t^2\right)$, $0 \leq t \leq \sqrt{3}$.

Solution. The length is

$$L = \int_0^{\sqrt{3}} \sqrt{(3t^2)^2 + (3t)^2}dt = \int_0^{\sqrt{3}} \sqrt{9t^2(t^2+1)}dt = \int_0^{\sqrt{3}} 3t\sqrt{t^2+1}dt = \left(t^2+1\right)^{\frac{3}{2}}\Big|_0^{\sqrt{3}} = 7$$

where the integration uses the substitution $u = t^2 + 1$, $du = 2t \cdot dt$.

Example. A cycloid is a curve formed by marking a point on a circle and tracing the curve it forms as it is rolled along a line. It can be shown that if the curve starts at the origin and the circle has radius 1, the cycloid is parametrized by $(x, y) = (t - \sin t, 1 - \cos t)$. Find the length of one arch of the cycloid ($0 \leq t \leq 2\pi$).

Solution. The length is

$$\begin{array}{rcll} L & = & \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2} dt & definition \\ & = & \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} dt & expand\, binomial \\ & = & \int_0^{2\pi} \sqrt{2 - 2\cos t} dt & trig\, identity \\ & = & \sqrt{2} \int_0^{2\pi} \sqrt{1-\cos t} dt & simplify \\ & = & \sqrt{2} \int_0^{2\pi} \sqrt{2\sin^2\left(\frac{t}{2}\right)} dt & power - reducing\, formula \\ & = & 2\int_0^{2\pi} \sin\left(\frac{t}{2}\right) dt & sine\, is\, positive\, on\ [0, \pi] \\ & = & -4\cos\left(\frac{t}{2}\right)\Big|_0^{2\pi} & integrate \\ & = & 8 & evaluate \end{array}$$

RECTANGULAR CURVES A rectangular function $y = f(x)$ can be parametrized as $(x, y) = (t, f(t))$. This converts the arclength formula to

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Example. Find the arclength of the semicircle $y = \sqrt{1-x^2}$ on $[-1, 1]$.

Solution. The length is

$$L = \int_{-1}^{1} \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2} dx = \int_{-1}^{1} \sqrt{\frac{1 - x^2 + x^2}{1-x^2}} dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = \arcsin x|_{-1}^{1} = \pi$$

Example. Find the arclength of $y = .8x^{1.25}$ on $[0, 9]$.

Solution. We find that

$$L = \int_0^9 \sqrt{1 + (x^{.25})^2} dx = \int_0^9 \sqrt{1 + \sqrt{x}} dx.$$

Using the substitution $u = 1 + \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$ with resubstitution $dx = 2(u-1)\,du$,

$$L = \int_1^4 2(u-1)\sqrt{u} du = 2\int_1^4 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du = 2\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right)\Big|_1^4 = \frac{232}{15}.$$

Rather than find the length of a curve between two particular points, we may wish to find a formula for the length between a particular point and an arbitrary point. The arclength function $L(x)$ of $y = f(x)$ starting at $x = a$ is

$$L(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\,dt.$$

Example. Find the arclength function of the catenary curve $y = \frac{1}{2}(e^x + e^{-x})$ starting at $x = 0$.

Solution. We first note that

$$1 + (y')^2 = 1 + \left(\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right)^2 = 1 + \frac{1}{4}e^{2x} - \frac{1}{2} + \frac{1}{4}e^{-2x} = \frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x} = \left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right)^2$$

Then

$$L(x) = \int_0^x \sqrt{\left(\frac{1}{2}e^t + \frac{1}{2}e^{-t}\right)^2}\,dt = \int_0^x \left(\frac{1}{2}e^t + \frac{1}{2}e^{-t}\right)dt = \frac{1}{2}e^t - \frac{1}{2}e^{-t}\Big|_0^x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

Example. Find the arclength function of the parabola $y = x^2$ starting at $x = 0$.

Solution. We see

$$L = \int_0^a \sqrt{1 + (2x)^2}\,dx = \frac{1}{2}\int \sec^3\theta\,d\theta$$

using the trig substitution $x = \frac{1}{2}\tan\theta$, $dx = \frac{1}{2}\sec^2\theta\,d\theta$, $\sqrt{1+4x^2} = \sec\theta$. We use integration by parts on the latter integral: $u = \sec\theta$, $du = \sec\theta\tan\theta\,d\theta$, $dv = \sec^2\theta\,d\theta$, $v = \tan\theta$.

$$\begin{aligned}
\int \sec^3\theta\,d\theta &= \sec\theta\tan\theta - \int \tan\theta\sec\theta\tan\theta\,d\theta \\
&= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta\,d\theta \\
&= \sec\theta\tan\theta + \int \sec\theta\,d\theta - \int \sec^3\theta\,d\theta \\
2\int \sec^3\theta\,d\theta &= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|
\end{aligned}$$

Undoing the trig substitution, we see

$$L = \frac{1}{4}2x\sqrt{1+4x^2} + \frac{1}{4}\ln\left|\sqrt{1+4x^2} + 2x\right|\Big|_0^a = \frac{1}{2}a\sqrt{1+4a^2} + \frac{1}{4}\ln\left(\sqrt{1+4a^2} + 2a\right).$$

Thus finding the arclength for a “simple” curve turns out to be quite complicated, involving trig substitution and integration by parts.

What if we try to find the arclength of the cubic $y = \frac{1}{3}x^3$? The formula gives us

$$L = \int_0^a \sqrt{1 + (x^2)^2}\,dx = \int_0^a \sqrt{1 + x^4}\,dx.$$

This function has no elementary antiderivative. Arclengths on it must be approximated by numerical methods. Many common functions and curves, such as x^n for most n, $\sin x$, $\cos x$, $\tan x$, and noncircular ellipses, lead to integrals with no elementary antiderivative.

EXERCISES. Find the arclength function of the following parametric curves starting at $t = 0$ (find the length when limits of integration are given).

	Name	Function	Hint
1	circle	$(3\sin t, 3\cos t)$, $[0, 2\pi]$	Trig Identity
2	spiral	$(t\cos t, t\sin t)$	Trig Substitution, Int. by Parts
3	logarithmic spiral	$(e^t\cos t, e^t\sin t)$	Trig Identity
4		$(\cos t, t + \sin t)$	Power Reducing
5	astroid	$\left(\cos^3 t, \sin^3 t\right)$, $\left[0, \frac{\pi}{2}\right]$	Trig Identity
6	involute	$(\cos t + t\sin t, \sin t - t\cos t)$	Trig Identity
7		$(5\cos t - \cos 5t, 5\sin t - \sin 5t)$	Sum Identity, Power Reducing
8	semicubical parabola	(t^2, t^3)	Factor, Substitution

Find the arclength function of the following rectangular curves starting at $x = 0$ (find the length when limits of integration are given).

	Name	Function	Hint
9	line	$y = mx + b$	Agrees with distance formula
10	logarithm	$y = \ln x$	Trig Substitution, Trig Identity
11	semicubical parabola	$y = x^{\frac{3}{2}}$	Substitution
12	astroid	$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$, $[0, 1]$	Tricky Algebra
13		$y = \frac{x^3}{6} + \frac{1}{2x}$	Perfect Square
14		$y = \frac{x^4}{8} + \frac{1}{4x^2}$	Perfect Square
15		$y = \frac{x^2}{4} - \frac{1}{2}\ln x$	Perfect Square
16		$y = \frac{x^{n+1}}{2(n+1)} + \frac{x^{-(n-1)}}{2(n-1)}$, $n \neq \pm 1$	Perfect Square
17		$y = \ln(\cos x)$	Trig Identity
18		$y = \ln(1 - x^2)$	Partial Fractions