THE BINOMIAL SERIES

The factorial function is defined recursively as $0! = 1$ and $n! = n \cdot (n-1)!$. We can think of $n!$ as multiplying together all the integers between 1 and n. Thus $n! = n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. Factorials are useful in discrete math, as $n!$ is the number of ways to arrange n distinct objects in order. Factorials are only defined for nonnegative integers, but they are still useful in calculus, where we deal with continuous functions. Recall the power rule, $\frac{d}{dx}x^n = nx^{n-1}$. If we repeatedly take derivatives, we find that

$$\frac{d^k}{dx^k}x^n = n(n-1) \cdot \ldots \cdot (n-k+1)\, x^{n-k} = \frac{n!}{(n-k)!}x^{n-k}$$

In particular, $\frac{d^n}{dx^n}x^n = n!$ when n is an integer.

A Taylor series attempts to represent a function $f(x)$ as an 'infinite polynomial' using a power series centered at 0:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \ldots$$

Evaluating at $x = 0$, we find $f(0) = a_0$. By differentiating repeatedly, we find $f^{(k)}(x) = \sum_{i=k}^{\infty} a_i \frac{i!}{(i-k)!} x^{i-k}$. Thus $f^{(k)}(0) = k! \cdot a_k$, so $a_k = \frac{f^{(k)}(0)}{k!}$, and the Taylor series is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

This explains why factorials often show up in Taylor series and related power series. Note that determining where these series converge usually requires the Ratio Test.

The binomial series is the Taylor series for $f(x) = (1+x)^n$ centered at 0. We have $f^{(k)}(x) = \frac{n!}{(n-k)!}(1+x)^{n-k}$, so $f^{(k)}(0) = \frac{n!}{(n-k)!}$. Thus the Taylor series is

$$(1+x)^n = \sum_{k=0}^{\infty} \frac{n!}{k!\,(n-k)!} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

where we define the binomial coefficient "n choose k" for n real, and $k \geq 0$ an integer as

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1) \cdot \ldots \cdot (n-k+1)}{k!}.$$

When n is a nonnegative integer, it can be shown that this counts the number of ways to choose k objects out of a set of n distinct objects.

If n is a nonnegative integer, the binomial series will be finite, terminating after $n+1$ terms. If not, it will be infinite. To determine where the binomial series converges, we use the Ratio Test. We find

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\binom{n}{k+1}x^{k+1}}{\binom{n}{k}x^k}\right| = \left|\frac{\frac{n!}{(k+1)!(n-k-1)!}}{\frac{n!}{k!(n-k)!}}x\right| = \left|\frac{k!\,(n-k)!}{(k+1)!\,(n-k-1)!}x\right| = \left|\frac{n-k}{k+1}x\right| \to |x|$$

The series will converge for $-1 < x < 1$ and diverge when $|x| > 1$. The endpoints must be tested separately. It can be shown that $\binom{n}{k} \approx \frac{C}{k^{1+n}}$, so comparison with a p-series shows that the binomial series converges at $x = -1$ exactly when $n \geq 0$ and it converges at $x = 1$ exactly when $n > -1$.

Example. Find the binomial series for $\sqrt{1+x}$.

Solution. We have $n=\frac{1}{2}$. We find $\binom{1/2}{0}=\frac{(1/2)!}{(1/2)!0!}=1$, $\binom{1/2}{1}=\frac{(1/2)!}{(-1/2)!1!}=\frac{1}{2}$, $\binom{1/2}{2}=\frac{(1/2)!}{(-3/2)!2!}=\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}=-\frac{1}{8}$, $\binom{1/2}{3}=\frac{(1/2)!}{(-5/2)!3!}=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{6}=\frac{1}{16}$, and in general $\binom{1/2}{k}=(-1)^{k+1}\frac{(2k-3)\cdots 5\cdot 3\cdot 1}{2^k k!}=\frac{(-1)^{k+1}(2k)!}{4^k (k!)^2 (2k-1)}$. The binomial series in this case is

$$\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+...$$

The series converges when $-1\le x\le 1$. Thus $\sqrt{2}=1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16}-\frac{5}{128}+...$

Example. Find the Taylor series for $\arcsin x$ centered at $x=0$.

Solution. We first find the series for the derivative $\frac{d}{dx}\arcsin x=\frac{1}{\sqrt{1-x^2}}$. Since $\binom{-1/2}{k}=\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2k-1}{2}\right)\frac{1}{k!}=\frac{(-1)^k(2k)!}{4^k(k!)^2}$, we see

$$(1+x)^{-\frac{1}{2}}=\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{4^k(k!)^2}x^k=1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+...$$

$$\left(1-x^2\right)^{-\frac{1}{2}}=\sum_{k=0}^{\infty}\frac{(2k)!}{4^k(k!)^2}x^{2k}=1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+...$$

$$\arcsin x=\sum_{k=0}^{\infty}\frac{(2k)!}{4^k(k!)^2(2k+1)}x^{2k+1}=x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7+...$$

THE BINOMIAL THEOREM The finite binomial series can be derived another way. Consider the expression $(x+y)^n$ (n a positive integer). We can use the distributive property to completely expand it out. Each term of the resulting sum has one factor being either x or y from each of the n binomials. For example,

$$(x+y)^3=xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy=x^3+3x^2y+3xy^2+y^3.$$

Thus there will be 2^n terms, some of which will be equivalent. The term x^ky^{n-k} will occur every time exactly k of the x's are chosen from the n binomials. Thus it will occur $\binom{n}{k}$ times. This shows that

$$(x+y)^n=\sum_{k=0}^{\infty}\binom{n}{k}x^ky^{n-k}.$$

This is the Binomial Theorem. Substituting 1 for y gives the finite binomial series. The binomial coefficients can be arranged as follows, where the rows give increasing values of n, and the diagonals give increasing values of k.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

This is called Pascal's Triangle. It can be simply generated by noting that each number in the interior of the triangle is the sum of the two numbers diagonally above it. For example, $10=4+6$. This works because of the identity $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, which can be proved using algebra or a counting argument.