MIXTURE PROBLEMS WITH EQUAL INFLOW AND OUTFLOW

In a mixture problem, there is a container with fluid flowing into and out of it. Some amount of a chemical is dissolved in it. The differential equation governing this situation will be

$$\begin{matrix} rate\,of\,change\,of \\ amount\,in\,container \end{matrix} = \begin{matrix} rate\,that \\ chemical\,arrives \end{matrix} - \begin{matrix} rate\,that \\ chemical\,departs \end{matrix}$$

Let $y(t)$ be the amount of the chemical in the container at time t. Let $V(t)$ be the volume of liquid in the container. We assume that inflow and outflow are equal, which implies that the volume is constant. Then

$$departure\,rate = \frac{y(t)}{V} \cdot outflow$$

Thus the differential equation is

$$\frac{dy}{dt} = arrival\,rate - \frac{y(t)}{V} \cdot outflow$$

A differential equation of the form $\frac{dy}{dt} = r + ky$ is separable. Separating, we see

$$\int \frac{dy}{r+ky} = \int dt$$

$$\frac{1}{k} \ln|r+ky| = t + C$$

$$r + ky = \pm e^{k(t+C)} = \pm e^{C} e^{kt} = Ae^{kt}$$

where $A = \pm e^{kC}$. Then

$$y = \frac{1}{k}\left(Ae^{kt} - r\right) = Be^{kt} - \frac{r}{k}$$

where $B = \frac{A}{k}$.

Example. In an oil refinery, a storage tank contains 2000 gallons of gasoline that initially has 100 lb of an additive dissolved in it. Gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at the same rate. How much additive is in the tank 20 min after the pumping process begins?

Solution. We have $V = 2000$, arrival rate $2\frac{lb}{gal} \cdot 40\frac{gal}{min} = 80\frac{lb}{min}$, and departure rate

$$dep = \frac{y\,lb}{2000\,gal} \cdot 40\frac{gal}{min} = \frac{y}{50}\frac{lb}{min}.$$

Thus the differential equation is $\frac{dy}{dt} = 80 - \frac{y}{50}$. The solution has the form

$$y = Be^{-\frac{t}{50}} + 4000.$$

Substituting the initial condition $y(0) = 100$, we find $B = -3900$. Thus

$$y = 4000 - 3900e^{-.02t}$$

$$y(20) = 4000 - 3900e^{-.02 \cdot 20} \approx 1385.75\,lb$$

Example. Beginning at time $t = 0$, a solution containing 40 mg/l of a drug is added to a patient's bloodstream, containing five liters of blood, at a rate of .15 l/hr, and well-mixed blood is withdrawn at the same rate. How much of the drug is in the patient's blood at time t, and what happens to this amount as time grows large?

Solution. We have $V = 5$, arrival rate $40\frac{mg}{l} \cdot .15\frac{l}{hr} = 6\frac{mg}{hr}$, and departure rate

$$dep = \frac{y\,mg}{5\,l} \cdot .15\frac{l}{hr} = .03y\frac{mg}{hr}.$$

Thus the differential equation is $\frac{dy}{dt} = 6 - .03y$. The solution has the form

$$y = Be^{-.03t} + 200.$$

Substituting the initial condition $y(0) = 0$, we find $B = -200$. Thus

$$y = 200 - 200e^{-.03t}.$$

Thus $y \to 200$ as time grows large.

NEWTON'S LAW OF COOLING

Suppose that an object is placed in a medium with a different temperature. Then the object transfers energy to (or from) the medium. The object could be a hot cup of tea in a room, a tomato in a refrigerator, or a chunk of molten metal in a water tank. We assume that the temperature of the medium is constant, as either it is large enough that the effect of the object on it is negligible, or its temperature is being held constant.

Newton's Law of Cooling states that the rate of change of the temperature $T(t)$ of the object is proportional to the difference between its temperature and the temperature of the medium T_m. Expressed as a differential equation, we have

$$\frac{dT}{dt} = -k\left(T(t) - T_m\right).$$

Now $\frac{dT}{dt} = kT_m - kT(t)$ has a solution of the form $T(t) = Be^{-kt} - \frac{kT_m}{-k} = Be^{-kt} + T_m$. Plugging in the initial condition $T(0) = T_0$, we find $B = T_0 - T_m$. Thus the solution to Newton's Law of Cooling is

$$T(t) = T_m + (T_0 - T_m)\,e^{-kt}.$$

Example. Suppose that a soda cooled from 90 degrees to 85 degrees after 10 minutes in a refrigerator whose temperature was 40 degrees. How long would it take the soda to cool to 50 degrees?

Solution. Using Newton's Law of Cooling, we see

$$T(t) = 40 + 50e^{-kt}.$$

We plug in our data point and find $85 = 40 + 50e^{-k10}$. Solving for k, we find $e^{-k10} = \frac{45}{50} = \frac{9}{10}$, so $k = -\frac{1}{10}\ln\frac{9}{10}$. Now we plug in the temperature and find $50 = 40 + 50e^{-kt}$. Solving for the time, we find $e^{-kt} = \frac{10}{50} = \frac{1}{5}$, so

$$t = \frac{\ln\frac{1}{5}}{-k} = \frac{10\ln\frac{1}{5}}{\ln\frac{9}{10}} \approx 152.75\,min$$

Exercises.

1. A 200 gallon tank is full of distilled water. At time $t = 0$, a solution containing 0.5 lb/gal of concentrate enters the tank at the rate of 5 gal/min, and the well-stirred mixture is withdrawn at the same rate. How many pounds of concentrate will the tank contain at time t? What is the limit of this process?

Solution:

$$y = 100 - 100e^{-.025t}$$

2. A water reservoir is full of 10,000 m^3 of water. At time t = 0, there is 0.02 g/m^3 salt in the reservoir. Water containing 0.01 g/m^3 of concentrate enters the reservoir at the rate of 1000 m^3 per day and the well-stirred mixture is withdrawn at the same rate. How much salt will the reservoir contain at time t?

3. An executive conference room of a corporation contains 4500 ft^3 of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 ft^3/min. A ceiling fan keeps the air in the room well-circulated and the air leaves the room at the same rate of 0.3 ft^3/min. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

Solution:

$$t = \frac{4500}{-.3} \ln\left(\frac{-.0001}{.04} + 1\right)$$

4. Suppose that a soda cooled from 80 degrees to 75 degrees after 20 minutes in a refrigerator whose temperature was 5 degrees. How long would it take the soda to cool to 10 degrees?

5. Suppose that a potato warmed from 72 degrees to 105 degrees after 30 minutes in an oven whose temperature was 350 degrees. How long would it take the potato to warm to 165 degrees?