DERIVING AN EXPLICIT FORMULA FOR THE FIBONACCI NUMBERS

The Fibonacci numbers are defined by the difference equation

$$x_{k+2} = x_{k+1} + x_k$$

with $x_0 = 0$, $x_1 = 1$. Thus the first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89... To find a solution, substitute $x_k = r^k$ into the difference equation, obtaining

$$r^{k+2} = r^{k+1} + r^k$$

$$r^k\left(r^2 - r - 1\right) = 0$$

The zeros of $r^2 - r - 1 = 0$ are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Thus $\alpha^k = \left(\frac{1+\sqrt{5}}{2}\right)^k$ and $\beta^k = \left(\frac{1-\sqrt{5}}{2}\right)^k$ are solutions to the difference equation. Thus any linear combination of these solutions is also a solution, so the general solution which spans the set of all possible solutions is

$$x_k = A\left(\frac{1+\sqrt{5}}{2}\right)^k + B\left(\frac{1-\sqrt{5}}{2}\right)^k.$$

To find the particular solution with the given initial conditions, we substitute in $k = 0$ and $k = 1$, obtaining

$$A + B = 0$$

$$A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

Solving this system of linear equations, we find $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$. Thus

$$x_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$$

This is an explicit formula for all Fibonacci numbers!

Note further that $\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right) < \frac{1}{2}$ and $\left(\frac{1-\sqrt{5}}{2}\right)^k \to 0$ as $k \to \infty$, so

$$f_k \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k$$

Specifically, the k^{th} Fibonacci number is the integer closest to the quantity on this right. This also implies that the ratio of consecutive Fibonacci numbers

$$\frac{f_{k+1}}{f_k} \to \frac{1+\sqrt{5}}{2}.$$

This quantity is called the golden ratio. It is widely used in art and architecture, as it is believed to be aesthetically pleasing.