INFINITE SERIES AND NUMBER THEORY

Infinite Series. Infinite series are typically introduced in calculus II. Loosely speaking, an infinite series is the sum of an infinite sequence. More precisely, it is the limit of the sequence of partial sums. That is,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

Many basic properties of finite sums extend to infinite series. In particular,

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k, \qquad \sum_{k=1}^{\infty} c \cdot a_k = c \sum_{k=1}^{\infty} a_k, \qquad \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_k b_j = \sum_{k=1}^{\infty} a_k \sum_{j=1}^{\infty} b_j = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_k b_j$$

Geometric Series. A sequence of the form a, ar, ar^2, ... , ar^{n-1} is called geometric. The number a is the leading term, and r is the ratio. To find the sum S_n, multiply by r and then subtract.

$$S_n = \sum_{k=0}^{n} ar^k = a + ar + ... + ar^n$$

$$r \cdot S_n = \sum_{k=0}^{n} ar^{k+1} = ar + ar^2 + ... + ar^{n+1}$$

$$S_n - r \cdot S_n = a - ar^{n+1}$$

$$S_n = \frac{a\left(1 - r^{n+1}\right)}{1 - r}$$

This is the sum of a finite geometric series. If we take a limit as $n \to \infty$, we find the sum of an infinite geometric series is $S = \frac{a}{1-r}$ when $|r| < 1$ and it diverges otherwise.

Repeating Decimals. Every fraction has a decimal expansion that either terminates or repeats. Given a repeating decimal, we can represent it as a geometric series and find its sum. Let $s = .\overline{a_1 a_2 ... a_n}$. Then $10^n s = a_1 a_2 ... a_n.\overline{a_1 a_2 ... a_n}$. Subtracting, we find $10^n s - s = a_1 a_2 ... a_n$. Hence $s = \frac{a_1 a_2 ... a_n}{10^n - 1} = \frac{a_1 a_2 ... a_n}{99...9}$. This is the fractional form of the repeating decimal. For example, $.123123123... = \frac{123}{999} = \frac{41}{333}$. Note also that $.999... = \frac{9}{9} = 1$. These are just different representations for the same number.

Telescoping Series. Another special type of series is a telescoping series. In this type of series, there is cancellation in the intermediate terms, so the series 'telescopes'.

$$\sum_{k=1}^{n}(a_k - a_{k+1}) = (a_1 - a_2) + (a_2 - a_3) + ... + (a_n - a_{n+1}) = a_1 - a_{n+1}$$

If $a_n \to 0$ as $n \to \infty$, the corresponding infinite series converges to a_1.

Consider an example. Using partial fractions, we can show $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Thus

$$\sum_{k=1}^{\infty} \frac{1}{k\,(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \lim \frac{1}{k+1} = 1.$$

The Harmonic Series. A series can diverge even when the corresponding sequence converges to 0. The most basic example of this is the Harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. There are several ways to show this. Assume to the contrary that the sum $H = \sum_{k=1}^{\infty} \frac{1}{k}$ is finite. Let $H_o = \sum_{k=1}^{\infty} \frac{1}{2k-1}$ and $H_e = \sum_{k=1}^{\infty} \frac{1}{2k}$ be the sums of the odd and even terms of the Harmonic series. Then these sums are both finite and $H = H_o + H_e$. Now $H_e = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{2}H$ and $H_o > H_e = \frac{1}{2}H$. Thus $H = H_o + H_e > H_e + H_e = H$, a contradiction, so the Harmonic series must diverge.

The Euler Product Formula. Consider the following product, which is expanded over all prime numbers.

$$\prod_{p}^{\infty} \frac{1}{1 - \frac{1}{p}}$$

Each factor has the form of the sum of an infinite geometric series. Thus we can multiply out the infinite product.

$$\prod_{p}^{\infty} \frac{1}{1 - \frac{1}{p}} = \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ...\right)\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ...\right)\left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + ...\right)\cdots$$

Each term in the expanded product uses one term from each product. Each term is the reciprocal of a product of powers of primes. By the Fundamental Theorem of Arithmetic, each integer can be factored uniquely into primes. Thus each integer occurs exactly once in the denominator. Thus we obtain the Harmonic series, which diverges.

$$\prod_{p}^{\infty} \frac{1}{1 - \frac{1}{p}} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... = \sum_{k=1}^{\infty} \frac{1}{k}$$

The Riemann Zeta Function. By a very similar argument, it can be shown that

$$\prod_{p}^{\infty} \frac{1}{1-\frac{1}{p^s}} = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

which converges whenever $s > 1$ by the Integral Test. If we allow s to be a complex number, we obtain the Riemann zeta function.

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

The infamous Riemann Hypothesis conjectures the values of the complex zeros of this function. The Euler product explains why it is related to the distribution of the prime numbers.

Sum of Inverse Primes. Next we consider whether the infinite series $\sum \frac{1}{p}$ of inverse primes converges or diverges. We take a logarithm of the Harmonic series. Then

$$\infty = \ln\left(\sum_{k=1}^{\infty} \frac{1}{k}\right) = \ln\left(\prod_{p}^{\infty} \frac{1}{1-\frac{1}{p}}\right) = \sum_{p} \ln\left(\frac{p}{p-1}\right) = \sum_{p} \ln\left(1+\frac{1}{p-1}\right) < \sum_{p} \frac{1}{p-1} < 1 + \sum_{p} \frac{1}{p}$$

where $x \geq \ln(1+x)$ follows from the tangent line for e^x at $x = 0$, $e^x \geq 1 + x$. Thus $\sum \frac{1}{p}$ diverges.

Note the Prime Number Theorem implies that $p_n \approx n \cdot \ln n$, so

$$\sum_{p} \frac{1}{p_n} \approx \sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n} \approx \int_{2}^{\infty} \frac{dn}{n \cdot \ln n} = \ln \ln \infty = \infty$$

by the Integral Test. However, $p_n > n \cdot \ln n$, so this is not a proof.

Power Series. A power series is an infinite series $p(x) = \sum a_n (x-a)^n$ which can be thought of as an infinite polynomial. It will converge on an interval centered at a, perhaps for all real numbers. The Taylor series of a function $f(x)$ is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

The derivation of this formula is typically explained in calculus II. Integrals and derivatives of Taylor series provide additional power series of functions. The power series in the table below are typically derived in calculus II. By plugging in specific constants, we find infinite series for many famous irrational numbers. These can be truncated to provide arbitrarily close decimal approximations for them.

A Table of Infinite Series for Irrational Numbers.

Power Series	Plug in	Resulting Series
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$x = 1$	$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + ...$
$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x = 1$	$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + ...$
	$x = \frac{1}{\sqrt{3}}$	$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = \frac{1}{\sqrt{3}} \left(1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 9} - \frac{1}{7 \cdot 27} + ...\right)$
$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n}$	$x = 1$	$\ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...$
$\frac{1}{2} \ln \left\lvert \frac{x+1}{x-1} \right\rvert = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$	$x = \frac{1}{3}$	$\ln 2 = \frac{2}{3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)9^n} = \frac{2}{3} \left(1 + \frac{1}{3 \cdot 9} + \frac{1}{5 \cdot 81} + \frac{1}{7 \cdot 729} + ...\right)$
$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$	$m = \frac{1}{2}$	
$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n$	$x = 1$	$\sqrt{2} = \sum_{n=0}^{\infty} \frac{(2k-3)!!}{(2k)!!} = 1 + \frac{1}{2} - \frac{1}{8} + \frac{3}{48} - \frac{15}{384} + \frac{105}{3840} - ...$

Proof That e is Irrational. We can use the series for e to show that it is irrational.

Assume to the contrary that e is rational, so $e = \frac{a}{b}$, where a and b are integers with $b > 1$. Let

$$x = b! \left(e - \sum_{n=0}^{b} \frac{1}{n!} \right) = b! \left(\frac{a}{b} - \sum_{n=0}^{b} \frac{1}{n!} \right) = a\,(b-1)! - \sum_{n=0}^{b} \frac{b!}{n!}.$$

Now x is an integer, since $n \le b$ for each term. Now x is positive since

$$x = b! \left(\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{b} \frac{1}{n!} \right) = \sum_{n=b+1}^{\infty} \frac{b!}{n!} > 0.$$

Now

$$\frac{b!}{n!} = \frac{1}{(b+1)(b+2)\cdots(b+(n-b))} \le \frac{1}{(b+1)^{n-b}}$$

so employing a geometric series, we have

$$x = \sum_{n=b+1}^{\infty} \frac{b!}{n!} \le \sum_{n=b+1}^{\infty} \frac{1}{(b+1)^{n-b}} = \sum_{k=1}^{\infty} \frac{1}{(b+1)^k} = \frac{1}{b+1} \left(\frac{1}{1 - \frac{1}{b+1}} \right) = \frac{1}{b} < 1.$$

Thus x is an integer with $0 < x < 1$, a contradiction. Thus e is irrational.