# COUNTING DERANGEMENTS

A derangement is a permutation (rearrangement) of $n$ objects so that none of them has its original position. Let $D_n$ be the number of derangements of length $n$. It is obvious that $D_1 = 0$ and $D_2 = 1$. It is not hard to see that $D_3 = 2$, since 312 and 231 are the only derangements of length 3. For larger values of $n$, we need to count using the "Inclusion-Exclusion Principle".

There are a total of $n!$ permutations of $n$ objects. We must subtract out all the permutations that fix at least one element. There are $n$ elements that can be fixed, and the other elements can be permuted $(n-1)!$ ways. However, this double-counts the permutations that fix at least two points, so we must add them back in. There are $\binom{n}{2}$ ways to fix two elements and $(n-2)!$ ways to permute the others. Now each permutation that fixes at least three elements has been counted $1 - 3 + \binom{3}{2} = 1$ time, so these must be subtracted out. Proceeding similarly, we find

$$D_n = n! - n\,(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + ... + (-1)^n$$

$$D_n = \frac{n!}{0!\,(n-0)!}n! - \frac{n!}{1!\,(n-1)!}(n-1)! + \frac{n!}{2!\,(n-2)!}(n-2)! - ... + (-1)^n \frac{n!}{n!\,(n-n)!}$$

$$D_n = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!} = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

The probability that a given permutation is a derangement is

$$P_n = \frac{D_n}{n!} = \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

This last expression is a Taylor polynomial for $e^x$ evaluated at $x = -1$. Thus as $n$ goes to infinity, we have

$$P_n \to \frac{1}{e} \approx .3679.$$

In fact, the convergence is very quick, as $D_n$ is the integer nearest to $\frac{n!}{e}$ for all $n > 0$.

**Exercises.**

1. Use a counting argument (not the formula) to prove that $D_{n+1} = n\,(D_n + D_{n-1})$.
2. How many permutations of $n$ objects contain exactly one cycle of length one?