2-Tone Coloring of Joins and Products of Graphs

Allan Bickle

Department of Mathematics
Western Michigan University
1903 W. Michigan
Kalamazoo, MI 49008
email: allan.e.bickle@wmich.edu

Abstract
A 2-tone coloring of a graph assigns two distinct colors (a label) to each vertex with the restriction that adjacent vertices have no common colors, and vertices at distance two have at most one common color. The 2-tone chromatic number of a graph is the minimum number of colors in any 2-tone coloring. When considering the 2-tone chromatic number of the join of two graphs, we find that each part of the join cannot repeat any label. Adding this restriction leads to the definition of pair coloring. We determine basic bounds and exact values of the pair chromatic number for specific classes of graphs.

We also determine the 2-tone chromatic number for Cartesian products of some graphs. These results show some unexpected connections to Latin squares and perfect matchings.

1 Introduction
Classical vertex coloring of graphs has been generalized in many different ways. (See [6], [10] for basic concepts and results.) A set coloring assigns a set of \( k \) colors to each vertex so that adjacent vertices have no common colors [5]. A \( \text{distance } k \text{ coloring} \) requires that vertices within distance \( k \) receive distinct colors. A related notion is an \( L'(2, 1) \) labeling, in which every vertex receives a nonnegative integer label, with vertices at distance one having labels at least two apart and vertices at distance two having labels at least one apart [9].

These definitions have been generalized [8] to define a coloring that assigns \( k \) colors to each vertex with restrictions on which sets may appear within distance \( k \) of each other that are stronger the smaller the distance.
**Definition 1.** Let $G$ be a graph, $k, t \in \mathbb{N}$, $[k] = \{1, 2, ..., k\}$, and let $P_t([k])$ denote the set of $t$-element subsets of $[k]$. A function $f : V(G) \to P_t([k])$ is called a proper $t$-tone $k$-coloring (or sometimes just a $t$-tone coloring) of $G$ if $|f(u) \cap f(v)| < d(u, v)$ for all distinct vertices $u$ and $v$ of $G$. A graph is $t$-tone $k$-colorable if it has a proper $t$-tone $k$-coloring. The $t$-tone chromatic number of $G$, denoted by $\tau_t(G)$, is the smallest positive integer $k$ for which $G$ has a proper $t$-tone $k$-coloring.

This definition was suggested by Gary Chartrand. It was first studied in a class project of Nicole Fonger, Josh Goss, Ben Phillips, and Chris Segroves [8]. A paper by Ben Phillips and this author [4] was explored in more detail in a paper by Ben Phillips and this author [4]. A paper by Dan Cranston, Jaehoon Kim, and William Kinnersley [7] provides some improved upper bounds. A paper by Deepak Bal, Patrick Bennett, Andrzej Dudek, and Alan Frieze [1] determines the 2-tone chromatic number of the random graph.

Note that for $t = 1$, $\tau_1(G) = \chi(G)$, the usual chromatic number of a graph $G$. This paper shall be solely concerned with the 2-tone chromatic number.

We shall often call $f(v)$ the label associated with the vertex $v$ of the coloring $f$, and the elements of $f(v)$ will be called colors. Thus, in a 2-tone coloring, each vertex has a label of 2 distinct colors. Adjacent vertices have no common colors, and vertices distance two apart have at most one common color. When the context is clear, the label $\{a, b\}$ will be denoted $ab$. Vertices distance two apart are called second-neighbors. A 2-chord of a cycle is a pair of vertices of the cycle at distance two apart.

Some basic results are immediate. The 2-tone chromatic number exists for all graphs. If $H$ is a subgraph of $G$ then $\tau_2(H) \leq \tau_2(G)$. We have $\tau_2(K_n) = 2n$.

If $G$ has components $G_i$, then $\tau_2(G) = \max \tau_2(G_i)$.

Several concepts related to proper coloring will be useful. The $k$-core of a graph $G$ is the maximal induced subgraph $H \subseteq G$ such that $\delta(G) \geq k$, if it exists. The core number of a vertex is the largest value for $k$ such that $v$ is in the $k$-core. Let $D(G)$ be the degeneracy of $G$, the maximum $k$ such that $G$ has a subgraph $H$ with $\delta(H) \geq k$. A graph is $k$-degenerate if it has degeneracy at most $k$. It is immediate from the definition that $D(G) \leq \Delta(G)$. A deletion sequence of a graph is a sequence of vertices formed by successively deleting a vertex of minimum degree. A construction sequence is the reversal of a deletion sequence. (See [2], [3] for background on these concepts.)

The 2-tone chromatic number has been determined for many basic classes of graphs in [8] and [4]. For complete multipartite graphs,

$$\tau_2(K_{a_1, a_2, ..., a_r}) = \sum_{i=1}^{r} \left\lfloor \frac{1 + \sqrt{1 + 8a_i}}{2} \right\rfloor.$$ 

Hence for the nontrivial star, $\tau_2(K_{1, a}) = \left\lfloor \frac{5 + \sqrt{1 + 8a}}{2} \right\rfloor$. For a nontrivial tree $T$
with maximum degree $\Delta$, $\tau_2(T) = \left\lceil \frac{5+\sqrt{1+8\Delta}}{2} \right\rceil$.

Consider forming a graph whose vertices are the ten possible labels for a 2-tone 5-coloring and all possible edges are added. This graph is the Petersen graph. In fact, the Petersen graph can be defined with just this labeling. Thus its 2-tone chromatic number is five, so any subgraph of the Petersen graph is 2-tone 5-colorable. (See Figure 1.) This result can also be seen by noting that the Petersen graph is the complement of the line graph of $K_5$. Let $L_n = L(K_n)$, the complement of the line graph of $K_n$. Thus $L_5$ is the Petersen graph. For $n \geq 4$, we have $\tau_2(L_n) = n$ [4].

For the cycle $C_n$, $\tau_2(C_n) = \begin{cases} 6 & n = 3, 4, 7 \\ 5 & \text{else} \end{cases}$. If $\tau_2(G) = k$, we call a 2-tone $k$-coloring of $G$ a minimum coloring. The minimum colorings of $C_n$ are unique up to isomorphism of colors and graphs for $n = 3, 4, 5, 6, 8, 9$. That is, any 2-tone $k$-coloring with $k = \tau_2(C_n)$ can be converted to any other by renaming of the colors and taking an automorphism [4].

The theta graph $\theta_{i,j,k}$ is formed by taking paths of lengths $i$, $j$, $k$ and identifying them at their endvertices. It necessarily contains three cycles of lengths $a = i + j$, $b = i + k$, $c = j + k$. We will use $(i,j,k)$ for $\theta_{i,j,k}$. The theta graph $\theta_{1,2,2}$ has $\tau_2(\theta_{1,2,2}) = 7$, and $(3,3,3)$, $(3,3,5)$, $(3,3,6)$, $(4,4,4)$, $(4,4,5)$, and $(3,3,9)$, all have 2-tone chromatic number 6. For all other theta graphs, $\tau_2(\theta_{i,j,k}) = \max\{\tau_2(C_a), \tau_2(C_b), \tau_2(C_c)\}$, where $a = i + j$, $b = i + k$, $c = j + k$ [4].

Figure 1: A 2-tone 5-coloring of the Petersen graph.
A graph $G$ is $2$-tone $k$-critical if $\tau_2(G) = k$ and for any proper subgraph $H$ of $G$, $\tau_2(H) < k$. The following theorem proves useful in several later theorems.

**Theorem 2.** [4] Let $G$ be a nonempty graph with $u, v \in V(G)$ not adjacent and set $e = uv$. Then $\tau_2(G + e) - \tau_2(G) \leq 1$.

## 2 Upper Bounds

Upper bounds for for $2$-tone and $t$-tone coloring were studied in 2011 by Cranston, Kim, and Kinnersley [7]. We consider some slight improvements on their bounds.

**Theorem 3.** Let $G$ be a graph with degeneracy $k$ and maximum degree $\Delta = \Delta(G)$. Then $\tau_2(G) \leq 2k + \left\lceil \frac{1+\sqrt{9+8(2\Delta - \Delta - k^2)}}{2} \right\rceil$.

**Proof.** Color $G$ with a construction sequence. When colored, a vertex $v$ has $j \leq k$ neighbors already colored, which excludes $2j$ colors. Each of the $j$ neighbors has at most $\Delta - 1$ second-neighbors. Then $v$ has at most $\Delta - j$ uncolored neighbors, each of which has at most $k - 1$ colored second-neighbors. Thus there are at most $j(\Delta - 1) + (\Delta - j)(k - 1)$ second-neighbors already colored, which is maximized when $j = k$. Thus we need $r$ extra colors, where $\binom{j}{2} \geq 2\Delta k - \Delta - k^2 + 1 \geq j(\Delta - 1) + (\Delta - j)(k - 1) + 1$. Solving, we find $r \geq \frac{1+\sqrt{9+8(2\Delta - \Delta - k^2)}}{2}$. □

**Theorem 4.** Let $G$ be a graph with maximum degree $\Delta \geq 2$. Then $\tau_2(G) \leq 2\Delta - 1 + \left\lceil \frac{1+\sqrt{1+8\Delta(\Delta - 1)}}{2} \right\rceil$.

**Proof.** We may assume $G$ is connected and regular. Now $G - e$ is $\Delta - 1$-degenerate. As in the previous proof, color $G - e$ using a construction sequence. Each vertex has at most $\Delta - 1$ neighbors which exclude at most $2(\Delta - 1)$ colors. There are at most $(\Delta - 1)^2 + (\Delta - 2) = \Delta(\Delta - 1) - 1$ second-neighbors already colored, so we need $r$ extra colors, where $\binom{j}{2} \geq \Delta(\Delta - 1)$. Solving, we find $r \geq \frac{1+\sqrt{1+8\Delta(\Delta - 1)}}{2}$. Lastly, adding $e$ back requires at most one more color by Theorem 2. □

Note that [7] proved the upper bound $\tau_2(G) \leq 2\Delta + \left\lceil \frac{1+\sqrt{9+8\Delta(\Delta - 1)}}{2} \right\rceil \leq [2 + \sqrt{2}] \Delta$. The bound in the previous theorem is at least one better.

In [4], $2$-tone $5$-colorable cubic graphs are characterized. It was conjectured that
Conjecture 5. Let $G$ be a cubic graph. Then
a. $\tau_2(G) \leq 8$;
b. $\tau_2(G) \leq 7$ when $G$ does not contain $K_4$;
c. $\tau_2(G) \leq 6$ when $G$ does not contain $K_4 - e$.

Part a was proved in [7].

Theorem 6. [7] If $G$ is cubic, $\tau_2(G) \leq 8$.

It is not hard to check that part c holds for $C_n \times K_2$ and Mobius ladders.

Proposition 7. [8] Let $n \geq 3$. We have $\tau_2(C_n \times K_2) = 6$.

Proof. Since $C_i \times K_2$ contains $C_4$, $\tau_2(C_n \times K_2) \geq 6$. The tables below represent colorings of $C_i \times K_2$, for $i = 3, 4, 5$. Each cell represents a vertex, and neighboring cells and cells on opposite ends of a row are adjacent. The first two tables can be concatenated to build products of longer cycles and $K_2$.

\[
\begin{array}{ccc|ccc|ccc}
25 & 34 & 16 & 25 & 13 & 24 & 16 & 23 & 16 & 24 & 15 & 46 \\
\end{array}
\]

A Mobius ladder can be formed by adding edges between opposite vertices of an even cycle. It can also be considered as a product of a cycle and $K_2$ with a 'twist' in it, hence the name.

Proposition 8. Let $G$ be a Mobius ladder. Then $\tau_2(G) = 6$.

Proof. Any Mobius ladder contains $C_4$. Consider the following Mobius ladders, where the left and right edges are identified with a 'twist'. In the latter two, the columns with parentheses can be successively inserted one at a time.

\[
\begin{array}{ccc|ccc|ccc}
12 & 46 & 23 & 12 & (35) & 46 & (15) & 23 & (16) \\
45 & 13 & 56 & 45 & (26) & 13 & (24) & 56 & (34)
\end{array}
\]

\[
\begin{array}{ccc|ccc|ccc}
12 & (35) & 46 & 15 & 26 & 45 & 16 & 23 \\
45 & (26) & 13 & 24 & 35 & 12 & 34 & 56
\end{array}
\]

Larger Mobius ladders can be constructed by concatenating smaller ones an odd number of times to maintain the 'twist'.

Part c was refuted [7] by demonstrating that it fails for the Heawood graph, which does not contain $K_4 - e$ (and indeed has girth 6). Their proof uses a clever but somewhat involved contradiction. The following shorter proof analyzes the maximal independent sets of the Heawood graph.


Proof. Recall that the Heawood Graph $G$ is the incidence graph for the Fano Plane, or equivalently, the Steiner Triple System of order 7 (STS(7)). Hence it is bipartite and any two vertices in the same partite set have exactly one common neighbor.
Consider the maximal independent sets of $G$. Each partite set is a maximum independent set of size seven. A vertex in one partite set is adjacent to three in the other, so there can be four vertices from the other partite set in the maximal independent set. Call an independent set with four vertices in one partite set and one in the other a 1-4-set. Two vertices in one partite set have a total of five distinct neighbors in the other set. Hence there can be an independent set with two vertices in each partite set.

Suppose that $G$ is 2-tone 6-colorable. If there is a color class of size seven, then it is one of the partite sets, and each vertex requires a distinct color for its other color. If there is a color class of size six contained in one of the partite sets, then there are at most two color classes with two vertices in this partite set, and at least four more colors are needed.

Hence each color class has size at most five. But then the six color classes must have sizes 5,5,5,5,4,4. Restricted to one partite set, they have sizes 5, 4, 2, or 1. Hence there are at least two with size at least four. Now any two color classes can overlap on at most one vertex of a partite set, so there are two with size four, which must 1-4-sets.

Now WLOG any 1-4-set contains four vertices that do not contain a triangle of the STS(7). But no other 1-4-set can contain only one vertex of this set, since then it would contain a triangle. This is a contradiction.

A 2-tone 7-coloring of the Heawood graph appears in [7], verifying that it has 2-tone chromatic number 7. After learning of the falsification of part c, this author checked all 21 cubic graphs of order 10 and found that one of them also violates part c.

**Proposition 10.** Let $G$ be the graph formed by starting with two copies of $K_{2,3}$ and adding a matching between the vertices of degree two in the two $K_{2,3}$’s (see Figure 2). Then $G$ is 2-tone 7-critical.

**Proof.** Suppose $G$ has a 2-tone 6-coloring. Now $K_{2,3}$ is uniquely 6-colorable, and each partite set requires three distinct colors. Let A and B be the partite
sets of one \(K_{2,3}\) and \(C\) and \(D\) be the partite sets of the other \(K_{2,3}\), with \(A\) and \(C\) being those that have three vertices, and hence are joined by the matching. If there is a color in common between \(A\) and \(C\), then it must appear on two vertices in each set, and hence on adjacent vertices. If they have no common colors, then \(B\) and \(C\) use the same three colors. But then they have two common labels at distance two apart.

\(G\) has only two edge orbits. Considering \(G - e\) for one edge of each type, 2-tone 6-colorings are easily obtained. Hence \(G\) is 7-critical.

The existence of a cubic 7-critical graph is of interest, especially since no cubic graph is 6-critical [4]. It is also interesting that both counterexamples are bipartite.

3 Joins and Pair Coloring

For the 2-tone chromatic number of a join of graphs, we have the following partial results.

**Proposition 11.** For the join \(G + H\),

\[ \tau_2 (G + H) \geq \tau_2 (G) + \tau_2 (H). \]

If \(G\) and \(H\) have diameter at most 2, then this is an equality.

**Proof.** The inequality follows since no common color can be used in both factors \(G\) and \(H\) of the join. If \(G\) and \(H\) both have diameter at most 2, then so does \(G + H\). Therefore combining minimum colorings for \(G\) and \(H\) creates no conflict, so the bound is achieved.

The bound may not be exact because vertices at distance greater than two in \(G\) will have distance two in \(G + H\). The converse is false, as for example the wheel \(W_6 = C_6 + K_1\) achieves the bound even though the 6-cycle has diameter 3. Thus for any factor in a join, the vertices must have a 2-tone coloring with the additional restriction that each label must be distinct. This motivates the following definition.

**Definition 12.** A *pair k-coloring* of a graph \(G\) is a 2-tone \(k\)-coloring in which every label is distinct. A graph is *pair \(k\)-colorable* if it has a pair \(k\)-coloring. The *pair chromatic number* of \(G\), \(pc(G)\), is the smallest \(k\) for which it has a pair \(k\)-coloring.

Some results on the pair chromatic number are immediate. We have \(pc(G) \geq \tau_2 (G)\), and if \(diam \, (G) \leq 2\), then this is an equality. Hence it is an equality for almost all graphs. If \(H\) is a subgraph of \(G\), then \(pc(H) \leq pc(G)\). It is not
difficult to show that \( \text{pc}(G + e) - \text{pc}(G) \leq 1 \). It is also straightforward to see that \( \text{pc}(G + H) = \text{pc}(G) + \text{pc}(H) \).

A graph \( G \) is pair \( k \)-colorable if and only if it is contained in \( L_k \). Thus if \( n > \tbinom{k}{2} \), \( \text{pc}(G) > k \). Equivalently, \( \text{pc}(G) \geq \frac{1 + \sqrt{1 + 8n}}{2} \). Thus \( \text{pc}(\mathbb{K}_n) = \left\lceil \frac{1 + \sqrt{1 + 8n}}{2} \right\rceil \). This also implies that given \( n_1 = n(G) \) and \( n_2 = n(H) \),

\[
\tau_2(G + H) \geq \tau_2(K_{n_1,n_2}) = \sum_{i=1,2} \left\lceil \frac{1 + \sqrt{1 + 8n_i}}{2} \right\rceil .
\]

since \( G + H \) contains \( K_{n_1,n_2} \) as a subgraph. This bound appears to be good for sparse graphs, but it is unclear exactly when it is an equality.

**Theorem 13.** Let \( G \) have degeneracy \( k \leq n - 1 \). Then

\[
\left\lceil \frac{1 + \sqrt{1 + 8n}}{2} \right\rceil \leq \text{pc}(G) \leq 2 + \left\lceil \frac{1 + \sqrt{1 + 8(n-k)}}{2} \right\rceil .
\]

**Proof.** The lower bound has already been justified. Color \( G \) with a construction sequence. Each vertex \( v \) has \( j \leq k \) neighbors which exclude at most \( 2j \) colors. There are at most \( n-j-1 \) labels that have already been used on non-neighbors of \( v \). Thus we need \( r \) extra colors, where \( \tbinom{r}{2} \geq n-j \). Solving, we find \( r \geq \frac{1 + \sqrt{1 + 8(n-j)}}{2} \). Thus we need at most \( 2j + \left\lceil \frac{1 + \sqrt{1 + 8(n-j)}}{2} \right\rceil \) colors to label \( v \), which is maximized when \( j = k \). \(\square\)

The upper bound is attained for the graph \( K_k + \mathbb{K}_{n-k} \). Since forests are exactly the \( 1 \)-degenerate graphs, we have the following corollary.

**Corollary 14.** For a forest \( F \),

\[
\left\lceil \frac{1 + \sqrt{1 + 8n}}{2} \right\rceil \leq \text{pc}(F) \leq 2 + \left\lceil \frac{1 + \sqrt{1 + 8(n-1)}}{2} \right\rceil .
\]

Thus there are usually three possible values for the pair chromatic number of a forest, but there are only two for \( n = \tbinom{r}{2} + 1, r \geq 2 \). Note that stars attain the upper bound. Characterizing the trees that attain the upper bound may be possible, but distinguishing between the other two values appears difficult.

**Conjecture 15.** Let \( F \) be a forest with order \( n \) and let \( r \) be the smallest integer such that \( n \leq \tbinom{r}{2} + 1 \). Then \( \text{pc}(F) = \frac{1 + \sqrt{1 + 8(n-1)}}{2} \) if and only if \( \Delta(F) \geq \tbinom{r-1}{2} + 1 \).

The reverse direction of this conjecture is obvious, but the forward direction seems difficult. If there is a counterexample, it cannot have maximum degree \( \tbinom{r-1}{2} \).
Proposition 16. Let $F$ be a forest with order $n$ and let $r$ be the smallest integer such that $n \leq \binom{r}{2} + 1$. If $\Delta(F) = \binom{r-1}{2}$, then $pc(F) \leq 1 + \left\lceil \frac{1+\sqrt{1+8(n-r)}}{2} \right\rceil$.

Proof. The result is easily checked for $1 \leq n \leq 7$. Let $F$ be a forest with order $n \geq 8$ and let $r \geq 5$ be the smallest integer such that $n \leq \binom{r}{2} + 1$. Add edges if necessary to form a tree $T$ with the same maximum degree. Let $v$ be a vertex with degree $\binom{r-1}{2}$, which WLOG receives label 12. Then its $\binom{r-1}{2}$ neighbors must receive all possible labels from $\{3, 4, ..., r + 1\}$. Now $F$ has at most $r-1$ vertices remaining, and $2(r-1)$ labels left. Label the remaining vertices using a construction sequence.

If the $i^{th}$ vertex labeled is adjacent to a neighbor of $v$, then there are at least $2(r-1) - (i-1) - 4 = 2r - i - 5 \geq 2r - (r-1) - 5 = r - 4 > 0$ labels remaining. If the $i^{th}$ vertex labeled is not adjacent to a neighbor of $v$, then its neighbor $u$ must have the color 1 or 2, and either excludes $r-1$ possible labels. The other color on $u$ excludes one more label. Since $u$ uses one of the labels already excluded, the preceding $i-1$ vertices exclude at most $i-2$ labels. Thus there are at least $2(r-1) - (r-1) - 1 - (i-2) = r - i \geq r - (r-1) = 1$ labels remaining. Thus $r+1$ colors suffice to label $T$. \hfill $\Box$

We now present two variations on Theorem 13.

Corollary 17. Let $G$ have degeneracy $k \leq n - 1$. Then

$$\left\lfloor \frac{1 + \sqrt{1 + 8n}}{2} \right\rfloor \leq pc(G) \leq \max_{0 \leq j \leq k} \left\{ 2j + \left\lceil \frac{1 + \sqrt{1 + 8(n-j)}}{2} \right\rceil \right\}.$$

Proof. Color $G$ with a construction sequence. Let the $k$-core of $G$ have order $n_k$. If vertex $v$ is the $i^{th}$ vertex colored and $j$ is the core number of $v$, then neighbors of $v$ exclude at most $2j$ colors. There are at most $n_j - j - 1$ labels that have already been used on non-neighbors of $v$. Thus we need $r$ extra colors, where $\binom{r}{2} \geq n_j - j$. Solving, we find $r \geq \left\lceil \frac{1 + \sqrt{1 + 8(n_j - j)}}{2} \right\rceil$. Thus we need at most $2j + \left\lceil \frac{1 + \sqrt{1 + 8(n_j - j)}}{2} \right\rceil$ colors to label $v$, and we must take the maximum over all $j$. \hfill $\Box$

This may be an improvement when $D(G) < \Delta(G)$. For regular graphs, the following corollary is an improvement.

Corollary 18. Let $G$ be a connected graph with maximum degree $\Delta \geq 1$. Then

$$\left\lfloor \frac{1 + \sqrt{1 + 8n}}{2} \right\rfloor \leq pc(G) \leq 2\Delta - 1 + \left\lceil \frac{1 + \sqrt{1 + 8(n-\Delta+1)}}{2} \right\rceil.$$

Proof. Since $G$ is connected, $G - e$ is $\Delta - 1$-degenerate. By Theorem 13, $pc(G - e) \leq 2(\Delta - 1) + \left\lceil \frac{1 + \sqrt{1 + 8(n - (\Delta - 1))}}{2} \right\rceil$. Lastly, adding $e$ back requires at most one more color. \hfill $\Box$
Definition 19. A graph $G$ is pair $k$-critical if for any proper subgraph $H$ of $G$, $pc(H) < pc(G) = k$.

For small values of $k$, it is possible to list all such graphs, since these are also the critical forbidden subgraphs of $L_k$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>Pair $k$-Critical Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$K_1$</td>
</tr>
<tr>
<td>3</td>
<td>$K_2$</td>
</tr>
<tr>
<td>4</td>
<td>$K_4, K_2$</td>
</tr>
<tr>
<td>5</td>
<td>$K_7, P_3$</td>
</tr>
</tbody>
</table>

It is considerably more difficult to determine all pair 6-critical graphs, which are the critical forbidden subgraphs of $L_5$ (the Petersen graph). We consider this problem below. Note that for any $k$, there is a finite number of pair $k$-critical graphs.

Proposition 20. Any graph $G$ has finitely many critical forbidden subgraphs.

Proof. Let $G$ have order $n$. Then $K_{n+1}$ is a critical forbidden subgraph of $G$, so any other critical forbidden subgraph must have order at most $n$. There are finitely many such graphs, some subset of which are not subgraphs of $G$. Some subset of these are critical.

Aside from the connection to 2-tone coloring, there is another reason why pair colorings are interesting. Consider labeling the vertices of a complete graph $K_n$ with 1 to $n$. Then each edge can be labeled with the pair of labels of its vertices. Each possible label occurs exactly once. Thus a pair $k$-coloring of a graph corresponds to a (usually different) edge-induced subgraph of $K_k$. Thus we can transform a question on pair $k$-coloring of a disconnected graph into a question on packing (or decomposition) of a complete graph. As packings have been widely studied, results on them can be applied to pair coloring.

Some examples of graphs and corresponding subgraphs for their minimum colorings are given in the following table.

<table>
<thead>
<tr>
<th>Graph</th>
<th>Corresponding Subgraph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_n$</td>
<td>$nK_2$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>$2P_3$</td>
</tr>
<tr>
<td>$C_5$</td>
<td>$C_5$</td>
</tr>
<tr>
<td>$C_6$</td>
<td>$K_{2,3}$</td>
</tr>
<tr>
<td>$C_8$</td>
<td>$W_4$</td>
</tr>
<tr>
<td>$C_9$</td>
<td>$K_5 - e$</td>
</tr>
<tr>
<td>$P_4$</td>
<td>$P_5$</td>
</tr>
<tr>
<td>$L_n$</td>
<td>$K_n$</td>
</tr>
<tr>
<td>$K_{r+s}(2)$</td>
<td>$K_r \cup K_s$</td>
</tr>
</tbody>
</table>
For example, consider a union of complete graphs. Each clique must have no common colors on its vertices. Thus its coloring corresponds to a matching in the complete graph. We employ the following lemma.

Lemma 21. Let $a_1, ..., a_k$ be integers with $1 \leq a_1 \leq ... \leq a_k \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $\sum a_i \leq (\frac{n}{2})^2$. Then $K_n$ has a packing with matchings of sizes $a_i$.

Proof. It is well-known that there is a decomposition $D$ of $K_n$ into $n$ or $n-1$ matchings of size $\left\lfloor \frac{n}{2} \right\rfloor$. Form a packing $B$ of $K_n$ with matchings $B_i$ of sizes $0 \leq b_1 \leq ... \leq b_k \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $\sum a_i = \sum b_i$ from $D$ by adding the necessary number of "empty matchings" and deleting the necessary number of edges. If $a_i \neq b_i$ for some $i$, there must be integers $i < j$ such that $b_i < a_i \leq a_j < b_j$. Form a subgraph $H$ by merging matchings $B_i$ and $B_j$ together. Each component of $H$ must be a path or even cycle. Since the sizes of $B_i$ and $B_j$ were unequal, $H$ must have a component path of odd length. Swap the edges on this path between $B_i$ and $B_j$. This modified packing has $k$ matchings whose sizes are closer to the integers $a_1, ..., a_k$ than before. (That is, the number $\sum |b_i - a_i|$ has been reduced by 2.) We can apply this process repeatedly until we produce a packing with matchings of sizes $a_i$.

Theorem 22. Let $a_1, ..., a_k$ be integers with $1 \leq a_1 \leq ... \leq a_k$ and $n = \sum a_i$. Then $pc(\bigcup_i K_{a_i}) = \max \left\{ 2a_k, \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil \right\}$.

Proof. Both lower bounds are immediate. Let $N = \max \left\{ 2a_k, \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil \right\}$. By the lemma, $K_N$ can be packed with matchings of sizes $a_1, ..., a_k$. Hence a pair $N$-coloring of $\bigcup_i K_{a_i}$ exists.

Corollary 23. Let $G$ be a disconnected graph with components $G_i$ with orders $a_1, ..., a_k$, $1 \leq a_1 \leq ... \leq a_k$. Then $\max \left\{ pc(G_i), \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil \right\} \leq pc(G) \leq \max \left\{ 2a_k, \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil \right\}$.

Hence graphs for which the lower inequality is strict are of interest. Such a graph must be disconnected with pair chromatic number at least 6. Four such critical graphs are listed in the table below. They have pair chromatic number at least 6 since (1) none of these graphs are subgraphs of the Petersen graph and (2) the corresponding decompositions do not pack $K_5$.

<table>
<thead>
<tr>
<th>Graph</th>
<th>Corresponding Decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{3,3} \cup P_3$</td>
<td>${K_1 + 2K_2, P_3 \cup K_2}$</td>
</tr>
<tr>
<td>$C_6 \cup 2K_2$</td>
<td>${K_{2,3}, 2K_2, 2K_2}$</td>
</tr>
<tr>
<td>$C_5 \cup K_{1,3}$</td>
<td>${C_5, K_3 \cup K_2}$</td>
</tr>
<tr>
<td>$2K_{1,3}$</td>
<td>${K_3 \cup K_2, K_3 \cup K_2}$</td>
</tr>
</tbody>
</table>
Now we can determine all pair 6-critical graphs. We define some notation. Let $T_{a-b-...-c}$ be a tree (caterpillar) with a spine having vertices of degree $a, b, \ldots, c$. Similarly, $T_{P_a-P_b-...-P_c}$ is formed by appending paths of lengths $a, b, \ldots, c$ to a path.

**Theorem 24.** There are exactly 11 pair 6-critical graphs, namely $K_{11}$, $K_{1,4}$, 2$K_{1,3}$, $T_{3-3} \cup P_3$, $T_{3-P_4-3}$, $K_3$, $C_4$, $C_7$, $C_{10}$, $C_6 \cup 2K_2$, $C_5 \cup K_{1,3}$.

**Proof.** [sketch] We consider $L_5$ (the Petersen graph). Certainly $K_{11}$ is pair 6-critical, so any other such graph has order at most 10. Now $K_{1,4}$ is pair 6-critical, so any other such graph has maximum degree at most 3.

Consider forests. We have 2$K_{1,3}$ 6-critical since any two vertices of the Petersen graph have distance at most two apart. Since the Petersen graph has a Hamiltonian path, no path is 6-critical. Checking cases shows that no spider (tree with exactly one vertex of degree three) is 6-critical. Checking cases when there are exactly two vertices of degree three at distance two apart does not find any 6-critical graph. When there are two adjacent vertices of degree 3, we find $T_{3-3} \cup P_3$ is 6-critical. Otherwise, checking cases shows that $T_{3-P_4-3}$ is the only other 6-critical forest.

Certainly the only 6-critical cycles are $K_3$, $C_4$, $C_7$, $C_{10}$. Now consider graphs containing a single cycle. The cycle $C_9$ is not the basis for any 6-critical unicyclic graph. Checking cases for $C_8$ does not produce any new 6-critical graphs. For $C_6$, we first find the disconnected graph $C_6 \cup 2K_2$. Checking cases for connected 6-critical unicyclic graphs does not find any more. Starting with $C_5$, we see $C_5 \cup K_{1,3}$ is a disconnected 6-critical graph. Checking cases when we consider appending one, two, three, four, or five trees does not produce any new 6-critical graphs.

There are two 2-tone 6-critical theta graphs ($\theta_{3,3,3}$ and $\theta_{3,3,5}$) with at most order 10, but both contain 2$K_{1,3}$. Checking cases shows that no graph formed by appending trees to a theta graph is 6-critical.

Consider adding another path to a theta graph. There are two possibilities. One produces two disjoint cycles, the other produces a subdivided $K_4$. Note first that any 6-critical graph can be produced by adding an edge between nonadjacent vertices of the Petersen graph and then deleting some number of edges. It is easily seen that this number must be at least four since adding the edge creates a 3-cycle, two 4-cycles, several 7-cycles, and two vertices of degree 4 that must be disrupted. Now two disjoint cycles must be 5-cycles, but checking cases does not produce any new 6-critical graphs. Checking cases organized by length of the longest cycle in a subdivided $K_4$ also does not produce any new 6-critical graphs.

Finally consider disconnected graphs that are not forests and not unicyclic. Begin by deleting a subgraph of small order and then considering what other component could be added to make the graph not pair 5-chromatic, and whether the resulting graph is 6-critical. Deleting one, two, or three vertices
produces nothing new. Deleting four or more vertices eliminates all but at most one cycle. This exhausts the search.

We can determine the pair chromatic number of cycles.

**Theorem 25.** We have

\[
pc(C_n) = \begin{cases} 
5 & n = 5, 6, 8, 9 \\
6 & n = 3, 4, 7, 10 - 15 \\
\lceil 1 + \sqrt{1 + 8n/2} \rceil & n \geq 11
\end{cases}
\]

**Proof.** There are two relevant lower bounds to consider. First, \( pc(C_n) \geq \tau_2(C_n) \). This is exact for \( 3 \leq n \leq 9 \) since the unique minimum colorings for all but \( C_7 \) do not repeat a pair and \( C_7 \) has a minimum coloring that does not repeat a pair [4].

The second lower bound requires that every vertex of the cycle have a distinct pair, so \( pc(C_n) \geq 1 + \sqrt{1 + 8n/2} \). We now consider explicit colorings of cycles where no pair is repeated. First consider the following broken cycle.

\[-12 - (56) - 34 - 25 - (36) - 14 - 23 - 45 - (26) - 13 - (46 - 15) - 24 - 16 - 35 -\

Without the pairs in parentheses, we have a ten-cycle. The pairs in parentheses can be 'inserted' into the cycle, preserving the necessary properties. That is, we can subdivide some edges and assign previously unused pairs to the new vertices. The two pairs in a single set of parentheses must be inserted at the same time. This provides constructions up to the 15-cycle, for which all pairs formed from six colors are used.

We next use induction on \( r \) to prove the existence of constructions for larger values of \( n \). Assume that for \( r \geq 6 \), there exists a 2-tone coloring of the cycle with \( \binom{r}{2} \) vertices using \( r \) colors, so that each possible pair is used exactly once. We want to insert new pairs of colors in between some of the existing pairs. Allowing color \( r + 1 \) adds \( r \) new pairs to insert.

We model this situation with a bipartite graph as follows. One partite set is the \( r \) new pairs to be added. The other partite set is the \( \binom{r}{2} \) possible locations for insertion. An edge joins two vertices if the particular pair can be inserted in the particular location. We seek a maximum matching in this bipartite graph.

Since each pair is distinct, and each pair uses the new color \( r + 1 \), a pair can be inserted as long as the other color is not one of the four used on the vertices between which the new vertex will be inserted. Thus each vertex in the location partite set has degree \( r - 4 \). Each existing color is used \( r - 1 \) times on the cycle. Since the vertices on which it is used form an independent set, each excludes two locations, leaving \( \binom{r}{2} - 2 (r - 1) = \frac{r}{2} (r - 1) (r - 4) \) valid locations, so this is the degree of the vertices in this partite set.
Consider a subset $S$ of new pairs of order $s$, and let its neighborhood $N(S)$ have order $n$. Then $s \left[ \frac{1}{2} (r-1)(r-4) \right] \leq n(r-4)$, so $n \geq s$. Thus the bipartite graph satisfies Hall’s condition, so it has a maximum matching. Thus the new pairs can be successively inserted up to a cycle of length $(r+1)$. By induction, we have constructions for all $n \geq 15$.

Our constructions achieve one of the lower bounds in all but the case $n = 10$, for which our construction is one larger. The bound cannot be achieved in this case since the Petersen graph is non-hamiltonian.

A corollary on paths follows immediately.

**Corollary 26.** For $n \geq 3$,

$$pc(P_n) = \begin{cases} 5 & 3 \leq n \leq 10 \\ \left\lceil \frac{n+1+8n}{2} \right\rceil & n \geq 11 \end{cases}.$$

**Proof.** The path requires at least five colors, and enough so that every vertex has a distinct pair. For $n \geq 11$, breaking the cycle constructed in the proof of the theorem for wheels yields the appropriate construction. For smaller values, the appropriate construction exists because the Petersen graph has a hamiltonian path.

**Corollary 27.** For the wheel $W_n = C_n + K_1$ and the fan $F_n = P_n + K_1$, $n \geq 3$, we have

$$\tau_2(W_n) = \begin{cases} 7 & n = 5, 6, 8, 9 \\ \left\lceil \frac{n+1+8n}{2} \right\rceil & n \geq 11 \end{cases}.$$ $$\tau_2(F_n) = \begin{cases} 7 & 3 \leq n \leq 10 \\ \left\lceil \frac{n+1+8n}{2} \right\rceil & n \geq 11 \end{cases}.$$

4 2-tone Coloring of Cartesian Products

We now consider 2-tone coloring of cartesian products of graphs. We first consider upper bounds. Note that if $G_i$ has maximum degree $\Delta_i$ and degeneracy $k_i$, then $\Delta(G_1 \times G_2) = \Delta_1 + \Delta_2$ and $D(G_1 \times G_2) = k_1 + k_2$ [2]. We apply the technique of [7].

**Theorem 28.** Let $G_i$ have degeneracy $k_i = D(G_i)$ and maximum degree $\Delta_i = \Delta_i(G)$. Further, let $k = k_1 + k_2$ and $M = (2\Delta_1k_1 - \Delta_1 - k_1^2) + (2\Delta_2k_2 - \Delta_2 - k_2^2) + k_1\Delta_2$. Then $\tau_2(G_1 \times G_2) \leq 2k + \left\lceil \frac{1+\sqrt{9+8M}}{2} \right\rceil$.
Proof. Number the vertices of $G_1$ in increasing order according to a construction sequence. Number the vertices of $G_1 \times G_2$ lexicographically and arrange them in a grid. Consider a construction sequence of increasing lexicographic order. Color $G_1 \times G_2$ with this construction sequence, so each vertex has at most $k$ neighbors which exclude at most $2k$ colors. When colored, a vertex $v$ has at most $(2\Delta_1 k_1 - \Delta_1 - k_1^2) + (2\Delta_2 k_2 - \Delta_2 - k_2^2) + k_1 \Delta_2 = M$ second-neighbors already colored by the proof of Theorem 3. The first term refers to second-neighbors in the same row, the second to second-neighbors in the same column, and the third to second-neighbors whose row and column are both different from $v$'s. Thus we need $r$ extra colors, where $(\Delta^2) \geq M + 1$. Solving, we find $r \geq \frac{1 + \sqrt{b + 8M}}{2}$.

Theorem 29. Let graphs $G_1$ and $G_2$ have maximum degrees $\Delta_1 \geq \Delta_2 \geq 1$, $\Delta_1 \geq 2$, and let $\Delta = \Delta (G_1 \times G_2) = \Delta_1 + \Delta_2$. Then

$$\tau_2 (G_1 \times G_2) \leq 2\Delta - 1 + \left[ \frac{1 + \sqrt{17 + 8(\Delta^2 - \Delta - \Delta_2 - 2\Delta_2)}}{2} \right].$$

Proof. We may assume $G_1 \times G_2$ is connected and regular. Number the vertices of $G_1$ in increasing order according to a construction sequence. Number the vertices of $G_1 \times G_2$ lexicographically and arrange them in a grid. Consider a construction sequence of increasing lexicographic order. Delete the edge $e$ between the last two vertices of this sequence. Now $G_1 \times G_2 - e$ is $\Delta - 1$-degenerate. Coloring it using a construction sequence, each vertex has at most $\Delta - 1$ neighbors which exclude at most $2(\Delta - 1)$ colors. The largest number of second-neighbors already colored occurs on the last two vertices colored. They have at most $\Delta_1 (\Delta_1 - 1) + (\Delta_2 - 1)^2 + \Delta_1 \Delta_2 = (\Delta_1 + \Delta_2)^2 - (\Delta_1 + 2\Delta_2) - \Delta_1 \Delta_2 + 1 = \Delta^2 - \Delta - \Delta_2 - \Delta_1 \Delta_2 + 1$ second-neighbors already colored. The first term refers to second-neighbors in the same row, the second to second-neighbors in the same column, and the third to second-neighbors whose row and column are both different from $e$'s. Thus we need $r$ extra colors, where $(\Delta^2) \geq \Delta^2 - \Delta - \Delta_2 - \Delta_1 \Delta_2 + 2$. Solving, we find $r \geq \frac{1 + \sqrt{17 + 8(\Delta^2 - \Delta - \Delta_2 - 2\Delta_2)}}{2}$. Lastly, adding $e$ back requires at most one more color by Theorem 2.

Note that [7] proved the upper bound $\tau_2 (G) \leq 2\Delta + \left[ \frac{1 + \sqrt{9 + 8(\Delta^2 - 1)}}{2} \right] \leq \left[ 2 + \sqrt{2} \right] \Delta$. The improvement on this bound is near greatest when $\Delta_1 \Delta_2$ is largest (for fixed $\Delta$), namely when $\Delta_1 = \Delta_2 = \frac{1}{2} \Delta$. Then $\Delta^2 - \Delta - \Delta_2 - \Delta_1 \Delta_2 = \frac{3}{4} \Delta^2 - 2\Delta = \frac{3}{4} (\Delta - 1)^2 - \frac{3}{4}$, so

$$\left[ 1 + \sqrt{17 + 8\left( \frac{3}{4} (\Delta - 1)^2 - \frac{3}{4} \right)} \right] = \left[ \frac{1 + \sqrt{11 + 6(\Delta - 1)^2}}{2} \right] \leq \left[ \frac{\sqrt{6\Delta}}{2} \right].$$
We now consider specific classes of graph products.

**Theorem 30.** Let \( m \leq n \). Then

\[
\tau_2(K_m \times K_n) = \begin{cases} 
6 & m = n = 2 \\
2n & \text{else}
\end{cases}
\]

**Proof.** Let \( G = K_m \times K_n, m \leq n \). Now \( K_n \subseteq G \), so \( \tau_2(G) \geq 2n \). Now certainly \( \tau_2(K_2 \times K_2) = 6 \), and \( \tau_2(K_1 \times K_2) = 4 \). Furthermore, we see \( G \subseteq K_n \times K_n \), so the proof reduces to this case.

Now it is well-known that there exist at least two mutually orthogonal Latin squares for all \( n \) except 2 and 6. We construct a 2-tone coloring of \( K_n \times K_n \) by using numbers 1 to \( n \) for the first Latin square and \( n + 1 \) to \( 2n \) for the second Latin square. Now juxtapose them into a Graeco-Latin square. This can be viewed as a 2-tone coloring of \( K_n \times K_n \), where each cell is a vertex and each pair of vertices in the same row or column are adjacent.

This leaves us to find a 2-tone coloring of \( K_6 \times K_6 \) with 12 colors. The following table represents this coloring, completing the proof.

<table>
<thead>
<tr>
<th>AB</th>
<th>37</th>
<th>28</th>
<th>59</th>
<th>46</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>68</td>
<td>35</td>
<td>7A</td>
<td>1B</td>
<td>24</td>
</tr>
<tr>
<td>78</td>
<td>2B</td>
<td>40</td>
<td>13</td>
<td>5A</td>
<td>69</td>
</tr>
<tr>
<td>56</td>
<td>0A</td>
<td>17</td>
<td>48</td>
<td>29</td>
<td>3B</td>
</tr>
<tr>
<td>34</td>
<td>15</td>
<td>9B</td>
<td>26</td>
<td>70</td>
<td>8A</td>
</tr>
<tr>
<td>12</td>
<td>49</td>
<td>6A</td>
<td>OB</td>
<td>38</td>
<td>57</td>
</tr>
</tbody>
</table>

Note that a 2-tone coloring exists in this last case even though a Graeco-Latin square does not because we have 66 labels to choose from, rather than 36.

This result yields bounds on the 2-tone chromatic number of a cartesian product of graphs.

**Corollary 31.** Let \( G \) and \( H \) be nontrivial graphs, not both \( K_2 \), with orders \( r \) and \( s \), respectively. Then

\[
\max \{ \tau_2(G), \tau_2(H), 6 \} \leq \tau_2(G \times H) \leq \max \{ 2r, 2s \}.
\]

We can also consider products of paths.

**Proposition 32.** For the grid \( P_m \times P_n \), \( m, n \geq 2 \), we have \( \tau_2(P_m \times P_n) = 6 \).

**Proof.** Since the grid contains a \( C_4 \), \( \tau_2(P_m \times P_n) \geq 6 \). Tile the grid with the following block, where cells represent vertices and each pair of neighboring cells are adjacent. This defines a 2-tone coloring.

<table>
<thead>
<tr>
<th>36</th>
<th>15</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>34</td>
<td>16</td>
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<tr>
<td>14</td>
<td>26</td>
<td>35</td>
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</tbody>
</table>

\[ \square \]
The grid can be wrapped around on itself to form a product of cycles. The same coloring shows the following.

**Corollary 33.** Let \( i, j \) be positive integers. We have \( \tau_2(C_{3i} \times C_{3j}) = 6 \).

These problems suggest the problem of determining \( \tau_2(C_i \times C_j) \) for all \( i \) and \( j \). We have the following partial results.

**Theorem 34.** We have

\[
\tau_2(C_3 \times C_i) = \begin{cases} 
6 & i = 3, 6, 9, 11, 12, i \geq 14 \\
7 & i = 4, 5, 7 
\end{cases}
\]

with \( i = 10, 13 \) undecided between 6 and 7.

**Proof.** We have already seen the result for \( C_3 \times C_3 \). The following blocks of length 3 and 8 agree on two consecutive columns, so they can be concatenated to obtain the other values.

<table>
<thead>
<tr>
<th>14</th>
<th>56</th>
<th>23</th>
<th>14</th>
<th>56</th>
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<th>15</th>
<th>36</th>
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<tr>
<td>25</td>
<td>34</td>
<td>16</td>
<td>25</td>
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<td>15</td>
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</table>

For \( i = 4, 5, 7, \) 2-tone \( 7 \)-colorings are easily determined, and the first is included in the following theorem. Suppose that \( G = C_3 \times C_4 \) has a 2-tone 6-coloring. Then each color is used four times, since every color appears once in each column. Then each color appears twice in some row, necessarily distance two apart. Thus there is a pair of columns distance two apart with three such pairs of colors in the same rows. Clearly no two can be in the same row. This trio of colors must be rotated in the column in between. The trio of the other three colors must be rotated to all three positions over these three columns. But this leads to a contradiction.

For \( G = C_3 \times C_5 \), each color is used five times, and so occurs twice in two rows distance two apart. Thus there are twelve such pairs, so of the five pairs of columns distance two apart, one must have at least three such pairs of colors in the same rows. A contradiction follows as before.

For \( G = C_3 \times C_7 \), it is possible to show by an exhaustive search that there is no 6-coloring.

**Corollary 35.** For \( j = 3, 6, 9, 11, 12, \) and \( j \geq 14 \), \( \tau_2(C_{3i} \times C_j) = 6 \).

**Lemma 36.** If a product of cycles has a 2-tone 6-coloring, then no vertex has both its two neighbors in its row sharing a common color and its two neighbors in its column sharing a common color.

**Proof.** Let \( v \) be a vertex in a product of cycles \( G \). Suppose to the contrary that \( G \) has a 2-tone 6-coloring and \( v \) has both its two neighbors in its row sharing a common color and its two neighbors in its column sharing a common color. Let \( u \) and \( w \) be neighbors of \( v \) that are not in the same row or column, and let
their mutual neighbor other than \( v \) be \( x \). Then \( u, v, w, \) and \( x \) induce \( C_4 \), so \( u \) and \( w \) have a common color. Thus every pair of neighbors of \( v \) has a color in common, and clearly no two neighbors of \( v \) have the same label.

Form a graph \( H \) whose vertices are the four colors not used on \( v \), and edges represent the labels of the neighbors of \( v \). Then \( H \) has size four, and every pair of edges is adjacent. But this is impossible.

**Theorem 37.** Let \( i \geq 3 \). Then \( \tau_2 (C_4 \times C_i) = 7 \).

**Proof.** Consider a given row of \( G = C_4 \times C_i \). It is possible for a row to be 2-tone colored with six colors so that every vertex has the property that each two neighbors share no common colors if and only if its length is a multiple of three.

First suppose \( i \neq 0 \mod 3 \) and \( G \) has a 2-tone 6-coloring. Now every 4-cycle 2-tone colored with six colors has every pair of nonadjacent vertices having one common color. Thus \( G \) has some vertex \( v \) so that its pairs of neighbors in both its row and column each have a color in common. But by the previous lemma, this is impossible.

Now suppose \( i = 0 \mod 3 \), and \( G \) has a 2-tone 6-coloring. By the lemma, each row must have the property that each two neighbors share no common colors. Then each six colors appear in every three consecutive vertices of a row, and two rows distance two apart must have a color repeated in each of the columns. Then a contradiction follows as in the case of the graph \( G = C_3 \times C_4 \). Thus \( \tau_2 (C_4 \times C_i) \geq 7 \).

We now show that equality can be achieved. The following tables correspond to colorings of \( C_4 \times C_3, C_4 \times C_4, \) and \( C_4 \times C_5 \). Each entry represents a vertex and cells are adjacent if they are neighbors in a row or column, or on opposite ends of a row or column. The first two colorings can be concatenated to form products of \( C_4 \) and larger cycles since they agree on the first two columns.

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<td>16</td>
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</tbody>
</table>

**Lemma 38.** Let \( n \geq 5, n \neq 0 \mod 3 \). Then any 2-tone 6-coloring of \( C_n \) uses at least four 2-chords.

**Proof.** Case 1. Let \( n = 3r + 1 \). Then \( C_n \) needs \( 6r + 2 \) colors with repetition. A color class using no 2-chords can use at most \( r \) vertices. A color class with \( r + 1 \) vertices uses at least two 2-chords. Four classes of \( r \) vertices and two of \( r + 1 \) produce \( 6r + 2 \) colors. Using a color class of \( r + k \) colors would require at least \( 3k - 1 \) 2-chords, so there is no advantage to using a class of more than \( r + 1 \) vertices.
Case 2. Let \( n = 3r + 2 \). Then \( C_n \) needs \( 6r + 4 \) colors with repetition. A color class using no 2-chords can use at most \( r \) vertices. A color class with \( r + 1 \) vertices uses at least one 2-chord. Two classes of \( r \) vertices and four of \( r + 1 \) produce \( 6r + 4 \) colors. Using a color class of \( r + k \) colors would require at least \( 3k - 2 \) 2-chords, so there is no advantage to using a class of more than \( r + 1 \) vertices.

In both cases, at least four 2-chords are required.

**Proposition 39.** Let \((i, j) = (5, n), 5 \leq n \leq 19, n \not\equiv 0 \pmod{3}, \) or \((7, 7)\) or \((7, 8)\). Then \( \tau_2(C_i \times C_j) \geq 7 \).

**Proof.** Each 2-chord corresponds to the vertex between its ends. In each case, we show that if the graph has a 2-tone 6-coloring, more 2-chords are required than there are vertices in the product of cycles. Then by the pigeonhole principle, some vertex must use 2-chords in both its row and column. But by Lemma 36, this is impossible, so the 2-tone chromatic number is at least 7.

For \((i, j) = (5, n), 5 \leq n \leq 19, n \not\equiv 0 \pmod{3}, \) we have order \( 5n \) and at least \( 4n + 4 \cdot 5 > 5n \) 2-chords. For \((i, j) = (7, 7)\), we have order 49 and at least \( 4 \cdot 7 + 4 \cdot 7 = 56 \) 2-chords. For \((i, j) = (7, 8)\), we have order 56 and at least \( 4 \cdot 7 + 4 \cdot 8 = 60 \) 2-chords.

For hypercubes, we have \( \tau_2(Q_1) = 4, \tau_2(Q_2) = \tau_2(Q_3) = 6, \) and \( \tau_2(Q_4) = 7 \) since \( Q_1 = K_2, Q_2 = C_4, Q_3 = C_4 \times K_2, \) and \( Q_4 = C_4 \times C_4 \). It was shown in [8] by Josh Goss that the 5- and 6-cubes can be 8-colored. Hence \( 7 \leq \tau_2(Q_5) \leq \tau_2(Q_6) \leq 8 \).

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**References**


