ARCLENGTH AND INTEGRALS

To find the length of a straight line, you can use the distance formula, which comes from the Pythagorean Theorem. But finding the length of a curve requires an integral which comes from a Riemann sum based on the distance formula. If you have a curve defined by the parametric equation $(x, y) = (f(t), g(t))$, $a \leq t \leq b$, the length of the curve is given by

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

Calculating arclength leads to many interesting integrals which illustrate many different integration techniques. Here are some examples.

Name	Function	Hint
circle	$(\cos t, \sin t)$, $[0, 2\pi]$	Trig Identity
spiral	$(t \cos t, t \sin t)$	Trig Substitution, Int. by Parts
logarithmic spiral	$(e^t \cos t, e^t \sin t)$	Trig Identity
cycloid	$(t - \sin t, 1 - \cos t)$, $[0, 2\pi]$	Power Reducing
	$(\cos t, t + \sin t)$	Power Reducing
astroid	$\left(\cos^3 t, \sin^3 t\right)$, $\left[0, \frac{\pi}{2}\right]$	Trig Identity
involute	$(\cos t + t \sin t, \sin t - t \cos t)$	Trig Identity
	$(5 \cos t - \cos 5t, 5 \sin t - \sin 5t)$	Sum Identity, Power Reducing
semicubical parabola	(t^2, t^3)	Factor, Substitution
helix	$(\cos t, \sin t, t)$	Trig Identity
	$(e^t \cos t, e^t \sin t, e^t)$	Trig Identity
	$\left(t, \sqrt{3}t^2, 2t^3\right)$	Perfect Square

For vectors of the length more than 2, the natural generalization of the arclength formula is the following, which applies to the previous three examples.

$$L = \int_a^b \left\| \overrightarrow{f}'(t) \right\| dt.$$

A rectangular equation $y = f(x)$ can be parametrized as $(t, f(t))$. This converts the arclength formula to

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Name	Function	Hint
circle	$y=\sqrt{1-x^2}$, $[-1,1]$	Arcsine
line	$y=mx+b$	Agrees with distance formula
parabola	$y=x^2$	Trig Substitution, Int. by Parts
logarithm	$y=\ln x$	Trig Substitution, Trig Identity
semicubical parabola	$y=x^{\frac{3}{2}}$	Substitution
	$y=x^{1.25}$	Resubstitution
astroid	$x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$, $[0,1]$	Tricky Algebra
catenary	$y=\frac{1}{2}\left(e^x+e^{-x}\right)$	Perfect Square
	$y=\frac{x^{n+1}}{2(n+1)}+\frac{x^{-(n-1)}}{2(n-1)}$, $n\neq\pm1$	Perfect Square
	$y=\frac{x^2}{4}-\frac{1}{2}\ln x$	Perfect Square
	$y=\ln(\cos x)$	Trig Identity
	$y=\ln\left(1-x^2\right)$	Partial Fractions

Many common functions and curves lead to integrals with no elementary antiderivative. Such curves include x^n for most values of n, $\sin x$, $\cos x$, $\tan x$, and ellipses.

The arclength formula can also be converted for a polar equation $r=f(\theta)$ using $x=r\cos\theta=f(\theta)\cos\theta$ and $y=r\sin\theta=f(\theta)\sin\theta$:

$$L=\int_a^b\sqrt{[f(\theta)]^2+[f'(\theta)]^2}d\theta.$$

Name	Function	Hint
circle	$r=\cos\theta$, $[0,2\pi]$	Trig Identity
circle	$r=\sin\theta+\cos\theta$, $\left[-\frac{\pi}{4},\frac{3\pi}{4}\right]$	Trig Identity
cardioid	$r=1-\cos\theta$, $[0,2\pi]$	Power Reducing
cardioid	$r=1+\cos\theta$, $[0,2\pi]$	Power Reducing
spiral	$r=\theta$	Trig Substitution, Int. by Parts
spiral	$r=\theta^2$	Factor, Substitution
logarithmic spiral	$r=e^\theta$	Easy
parabola	$r=\frac{2}{1+\cos\theta}$	Factor, Power Reducing, Int. by Parts
	$r=\sin^2\theta$	Factor, Trig Identity
	$r=\cos^3\theta$	Factor, Trig Identity, Power Reducing
	$r=\sqrt{1+\sin 2\theta}$	Trig Identity