AMAZING ARCLENGTH

Dr. Allan Bickle

Western Michigan University

Math Club / Pi Mu Epsilon

January 18, 2013

History of Arclength

In ancient times, great thinkers considered it impossible to find the length of a curve.

Polygons inscribed in a circle used to approximate π.

Method of exhaustion

1645 - Logarithmic Spiral - Torricelli

1658 - Cycloid - Christopher Wren

1659 - Semicubical Parabola - William Neile

1691 - Catenary - Leibniz

Integral Form - codiscovered by

1659 - Hendrik van Heuraet

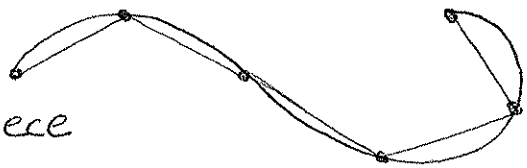
1660 - Pierre de Fermat

Source: Wikipedia

Deriving the Length of a Curve

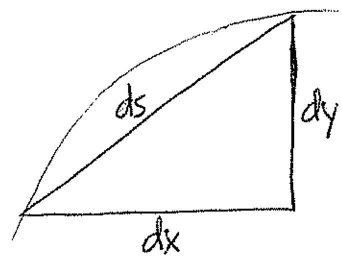
Begin by approximating the length of the curve.

Partion the curve.



Approximate each piece with a line segment.

Write the length ds in terms of dx and dy.



$$(ds)^2 = (dx)^2 + (dy)^2$$

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

Sum the pieces to obtain a Riemann sum.

Take a limit as the length of the pieces goes to 0. Obtain an integral.

$$\sum ds = \sum \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$L = \int ds = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

for parametrization $(x,y) = (f(t), g(t))$.

Rectangular Function $y = f(x)$

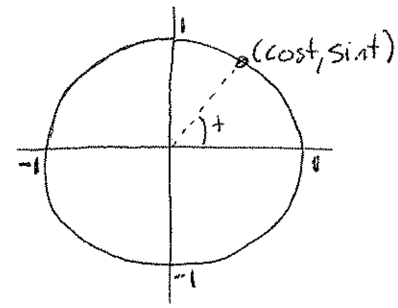
Parametrize it. $(x,y) = (t, f(t))$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Unit Circle

$$x^2 + y^2 = 1$$

$$(x,y) = (\cos t, \sin t) \quad t \in [0, 2\pi]$$



$$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$

$$= \int_0^{2\pi} dt$$

$$= t \Big|_0^{2\pi}$$

$$= 2\pi$$

Rectangular Form:

$$y = \pm\sqrt{1-x^2}$$

$$y' = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$L = 2\int_{-1}^{1} \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\, dx$$

$$= 2\int_{-1}^{1} \sqrt{1 + \frac{x^2}{1-x^2}}\, dx$$

$$= 2\int_{-1}^{1} \sqrt{\frac{1-x^2+x^2}{1-x^2}}\, dx$$

$$= 2\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx$$

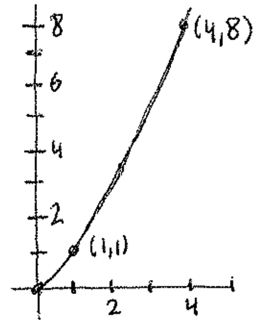
$$= 2 \arcsin x \Big|_{-1}^{1}$$

$$= 2\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= 2\pi$$

Semicubical Parabola

$$y = x^{\frac{3}{2}}$$



$$L = \int \sqrt{1+\left(\tfrac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx$$

$$= \int \sqrt{1+\tfrac{9}{4}x}\;dx$$

substitution $\quad u = 1+\frac{9}{4}x$

$\qquad\qquad\qquad du = \frac{9}{4}\,dx$

$$= \int \sqrt{u}\;\tfrac{4}{9}\,du$$

$$= \tfrac{4}{9}\,\tfrac{2}{3}\,u^{\frac{3}{2}} + C$$

$$= \tfrac{8}{27}\left(1+\tfrac{9}{4}\right)^{\frac{3}{2}} + C$$

Parametrization $\quad (x,y) = (t^2, t^3)$

$$L = \int \sqrt{(2t)^2 + (3t^2)^2}\;dt$$

$$= \int \sqrt{4t^2 + 9t^4}\;dt$$

$$= \int t\sqrt{4+9t^2}\;dt \qquad u = 4+9t^2, \quad du = 18t\,dt$$

$$= \int \sqrt{u}\;\tfrac{1}{18}\,du$$

$$= \tfrac{1}{18}\,\tfrac{2}{3}\,u^{\frac{3}{2}} + C$$

$$= \tfrac{1}{27}\left(4+9t^2\right)^{\frac{3}{2}} + C$$

Is this the same answer?

How about $y = x^{1.25}$?

"semisemiquintical parabola"?

$$L = \int \sqrt{1 + \left(\tfrac{5}{4}x^{.25}\right)^2}\,dx$$

$$= \int \sqrt{1 + \tfrac{25}{16}\sqrt{x}}\,dx$$

Substitution... $u = 1 + \frac{25}{16}\sqrt{x}$

$du = \frac{25}{32}\frac{1}{\sqrt{x}}\,dx$

$$= \int \tfrac{32}{25}\sqrt{x}\,\sqrt{u}\,du$$

Resubstitution! $\sqrt{x} = \frac{16}{25}(u-1)$

$$= \int \tfrac{32}{25}\,\tfrac{16}{25}(u-1)\sqrt{u}\,du$$

$$= \tfrac{512}{625}\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$

$$= \tfrac{512}{625}\left(\tfrac{2}{5}u^{\frac{5}{2}} - \tfrac{2}{3}u^{\frac{3}{2}}\right) + C$$

$$= \tfrac{1024}{3125}\left(1 + \tfrac{25}{16}\sqrt{x}\right)^{\frac{5}{2}} - \tfrac{1024}{9375}\left(1 + \tfrac{25}{16}\sqrt{x}\right)^{\frac{3}{2}} + C$$

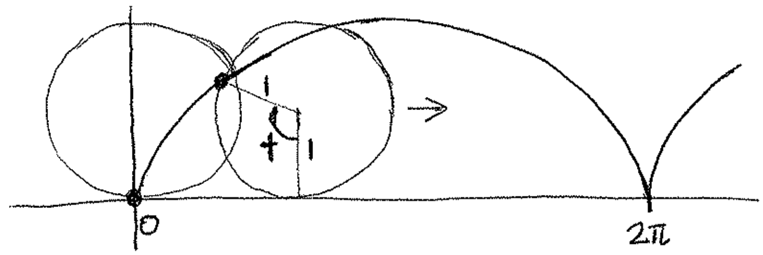
A similar approach works for

$y = x^{\frac{2n+1}{2n}}$, $n \in \mathbb{Z}^+$

Cycloid

Mark a point on a circle.

Roll it along a line, and trace the resulting curve. This is the cycloid.



It can be shown (ICBS) that the cycloid is parametrized by

$$(x, y) = (t - \sin t,\ 1 - \cos t)$$

$$L = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos t}\, dt$$

Use the power-reducing formula

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \Rightarrow 2\sin^2\left(\frac{t}{2}\right) = 1 - \cos t$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{2\sin^2\left(\tfrac{t}{2}\right)}\, dt$$

$$= 2 \int_0^{2\pi} \left|\sin\left(\tfrac{t}{2}\right)\right| dt$$

$$= 2\left(-2\cos\left(\tfrac{t}{2}\right)\right)\Big|_0^{2\pi}$$

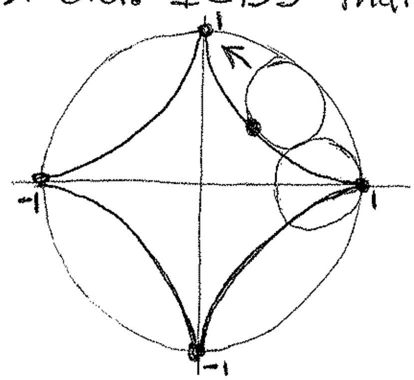
$$= -4(-1-1)$$

$$= 8$$

The Astroid

Mark a point on a circle and roll it around the inside of a larger circle. The resulting curve is a hypocycloid. If the inner radius is $\frac{1}{4}$ the outer one, the curve is the astroid. ICBS that the equations

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$



$$(x,y) = (\cos^3 t, \sin^3 t)$$

describe it.

$$(x', y') = (-3\cos^2 t \sin t, 3\sin^2 t \cos t)$$

$$= 3\sin t \cos t(-\cos t, \sin t)$$

$$L = 4\int_0^{\frac{\pi}{2}} 3\sin t \cos t \sqrt{(-\cos t)^2 + (\sin t)^2}\, dt$$

$$= 12\int_0^{\frac{\pi}{2}} \sin t \cos t \, dt \qquad \begin{array}{l} u = \sin t \\ du = \cos t \, dt \end{array}$$

$$= 12\left(\frac{1}{2}\sin^2 t\right)\Big|_0^{\frac{\pi}{2}}$$

$$= 6(1-0)$$

$$= 6$$

Alternative:

$$12\int_0^{\frac{\pi}{2}} \sin t \cos t \, dt$$

$$= 6\int_0^{\frac{\pi}{2}} \sin 2t \, dt$$

$$= 6\left(-\frac{1}{2}\cos 2t\right)\Big|_0^{\frac{\pi}{2}}$$

$$= -3(-1-1)$$

$$= 6$$

The Astroid (Rectangular)

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

$$y = \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

$$y' = \frac{3}{2}\sqrt{1 - x^{\frac{2}{3}}}\left(-\frac{2}{3}x^{-\frac{1}{3}}\right)$$

$$= -x^{-\frac{1}{3}}\sqrt{1 - x^{\frac{2}{3}}}$$

$$1 + (y')^2 = 1 + x^{-\frac{2}{3}}\left(1 - x^{\frac{2}{3}}\right) = x^{-\frac{2}{3}}$$

$$\sqrt{1 + (y')^2} = \sqrt{x^{-\frac{2}{3}}} = x^{-\frac{1}{3}}$$

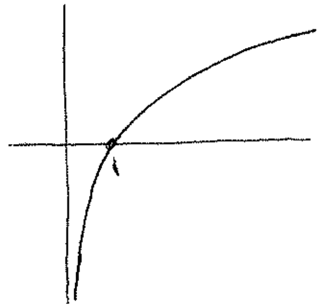
$$L = 4\int_0^1 x^{-\frac{1}{3}}\,dx$$

$$= 4\frac{3}{2}x^{\frac{2}{3}}\Big|_0^1$$

$$= 6$$

Natural Logarithm

$y = \ln x$



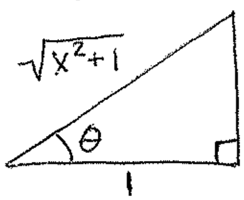
$y' = \frac{1}{x}$

$L = \int \sqrt{1 + \left(\frac{1}{x}\right)^2}\, dx$

$= \int \sqrt{\frac{x^2+1}{x^2}}\, dx$

$= \int \frac{\sqrt{x^2+1}}{x}\, dx$

Trig Substitution!



$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$\sqrt{x^2+1} = \sec\theta$

$= \int \frac{\sec\theta}{\tan\theta} \sec^2\theta\, d\theta$

An integral of the form $\int \sin^m x \cos^n x\, dx$...

$\frac{\sec^3\theta}{\tan\theta} = \frac{1}{\cos^3\theta}\frac{\cos\theta}{\sin\theta} = \frac{1}{\cos^2\theta \sin\theta}$

$= \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta \sin\theta} = \frac{\sin\theta}{\cos^2\theta} + \frac{1}{\sin\theta}$

$= \int \frac{\sin\theta}{\cos^2\theta}\, d\theta + \int \csc\theta \frac{\csc\theta + \cot\theta}{\csc\theta + \cot\theta}\, d\theta$

$u = \cos\theta \qquad u = \csc\theta + \cot\theta$

$du = -\sin\theta\, d\theta \qquad du = -\csc\theta\cot\theta - \csc^2\theta\, d\theta$

$= \int \frac{-du}{u^2} + \int \frac{-du}{u}$

$= \frac{1}{u} - \ln|u| + C$

$= \sec\theta - \ln|\csc\theta + \cot\theta| + C$

$= \sqrt{x^2+1} - \ln\left|\frac{\sqrt{x^2+1}}{x} + \frac{1}{x}\right| + C$

Note: $y = e^x$ is the same curve!

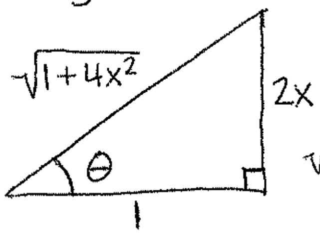
Evaluating the Integral $\int \sin^m x \cdot \cos^n x \, dx$

<table>
<tr><th></th><th></th><th colspan="4">Power of sin x</th></tr>
<tr><th></th><th></th><th>negative, even</th><th>negative, odd</th><th>nonnegative, even</th><th>positive, odd</th></tr>
<tr><td rowspan="6">Power of cos x</td><td>positive, odd</td><td>←</td><td>←</td><td>$\cos^2 x = 1-\sin^2 x$
$u = \sin x$
$du = \cos x$</td><td>← OR
↓</td></tr>
<tr><td rowspan="2">nonnegative, even</td><td colspan="2">If $n \geq -m$, let $\cos^2 x = 1-\sin^2 x$</td><td rowspan="2">Power-reducing
$\sin^2 x = (1-\cos 2x)/2$
$\cos^2 x = (1+\cos 2x)/2$</td><td rowspan="2">$\sin^2 x = 1-\cos^2 x$
$u = \cos x$
$du = -\sin x$</td></tr>
<tr><td>If $-m > n$
$\csc^2 x = 1+\cot^2 x$
$u = \cot x$
$du = -\csc^2 x$</td><td>If $-m > n$
$\cot^2 x = \csc^2 x - 1$
Integration by Parts
$dv = \csc^2 x$</td></tr>
<tr><td>negative, odd</td><td colspan="2" rowspan="3">multiply by $\cos^2 x+\sin^2 x$
break fraction, cancel functions</td><td>If $-n > m$
$\tan^2 x = \sec^2 x - 1$
Integration by Parts
$dv = \sec^2 x$</td><td>↓</td></tr>
<tr><td></td><td>If $m \geq -n$, let
$\sin^2 x = 1-\cos^2 x$</td><td></td></tr>
<tr><td>negative, even</td><td>If $-n > m$
$\sec^2 x = 1+\tan^2 x$
$u = \tan x$
$du = \sec^2 x$</td><td>↓</td></tr>
</table>

Parabola $y = x^2$

$$L = \int \sqrt{1 + (2x)^2}\, dx$$

Trig Substitution



$$x = \tfrac{1}{2}\tan\theta$$
$$dx = \tfrac{1}{2}\sec^2\theta\, d\theta$$
$$\sqrt{1+4x^2} = \sec\theta$$

$$= \frac{1}{2}\int \sec^3\theta\, d\theta$$

Integration by Parts

$u = \sec\theta \qquad dv = \sec^2\theta\, d\theta$

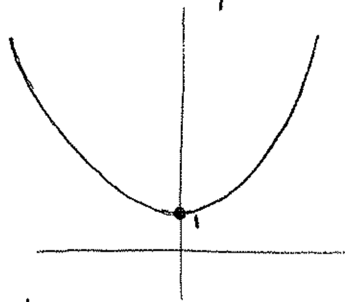
$du = \sec\theta\tan\theta\, d\theta \qquad v = \tan\theta$

$$\begin{aligned}
\int \sec^3\theta\, d\theta &= \sec\theta\tan\theta - \int \tan\theta\sec\theta\tan\theta\, d\theta \\
&= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta\, d\theta \\
&= \sec\theta\tan\theta + \int \sec\theta\, d\theta - \int \sec^3\theta\, d\theta \\
2\int \sec^3\theta\, d\theta &= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| \\
\int \sec^3\theta\, d\theta &= \tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C
\end{aligned}$$

$$\begin{aligned}
L &= \tfrac{1}{4}\sec\theta\tan\theta + \tfrac{1}{4}\ln|\sec\theta + \tan\theta| + C \\
&= \tfrac{1}{4}\sqrt{1+4x^2}\, 2x + \tfrac{1}{4}\ln\left|\sqrt{1+4x^2} + 2x\right| + C \\
&= \tfrac{1}{2}x\sqrt{1+4x^2} + \tfrac{1}{4}\ln\left|\sqrt{1+4x^2} + 2x\right| + C
\end{aligned}$$

Catenary

A hanging cable forms a catenary curve.



$$y = \frac{1}{2}(e^x + e^{-x})$$

$$y' = \frac{1}{2}(e^x - e^{-x})$$

$$L = \int \sqrt{1 + \left(\tfrac{1}{2}(e^x - e^{-x})\right)^2}\, dx$$

$$= \int \sqrt{1 + (\tfrac{1}{2}e^x)^2 - \tfrac{1}{2} + (\tfrac{1}{2}e^{-x})^2}\, dx$$

$$= \int \sqrt{(\tfrac{1}{2}e^x)^2 + \tfrac{1}{2} + (\tfrac{1}{2}e^{-x})^2}\, dx$$

$$= \int \sqrt{(\tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x})^2}\, dx$$

$$= \int \tfrac{1}{2}(e^x + e^{-x})\, dx$$

$$= \tfrac{1}{2}(e^x - e^{-x}) + C$$

Perfect Squares are useful for more...

$$y = \frac{x^{n+1}}{2(n+1)} + \frac{x^{-(n-1)}}{2(n-1)}, \quad n \neq \pm 1, \quad x \geq 0$$

$$y' = \tfrac{1}{2}x^n - \tfrac{1}{2}x^{-n}$$

$$L = \int \sqrt{1 + (\tfrac{1}{2}x^n - \tfrac{1}{2}x^{-n})^2}\, dx$$

$$= \int \sqrt{1 + (\tfrac{1}{2}x^n)^2 - \tfrac{1}{2} + (\tfrac{1}{2}x^{-n})^2}\, dx$$

$$= \int \sqrt{(\tfrac{1}{2}x^n + \tfrac{1}{2}x^{-n})^2}\, dx$$

$$= \int (\tfrac{1}{2}x^n + \tfrac{1}{2}x^{-n})\, dx$$

$$= \frac{x^{n+1}}{2(n+1)} - \frac{x^{-(n-1)}}{2(n-1)} + C$$

The same approach works for any function of the form

$$g(x) = \frac{1}{2}\left(\int f(x)\,dx - \int \frac{dx}{f(x)}\right), \quad f(x) \geq 0.$$

How about $y = \ln(1-x^2)$, $x \in (-1,1)$?

$$y' = \frac{-2x}{1-x^2}$$

$$1+(y')^2 = 1 + \left(\frac{-2x}{1-x^2}\right)^2 = \frac{(1-x^2)^2}{(1-x^2)^2} + \frac{4x^2}{(1-x^2)^2}$$

$$= \frac{1-2x^2+x^4+4x^2}{(1-x^2)^2} = \frac{(1+x^2)^2}{(1-x^2)^2}$$

$$L = \int \sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}}\,dx$$

$$= \int \frac{1+x^2}{1-x^2}\,dx$$

$$= \int \left(-1 + \frac{2}{1-x^2}\right)dx$$

Partial Fractions!

$$\frac{2}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$2 = A(1+x) + B(1-x)$$

$$x = 1 \Rightarrow A = 1$$

$$x = -1 \Rightarrow B = 1$$

$$= \int \left(-1 + \frac{1}{1-x} + \frac{1}{1+x}\right)dx$$

$$= -x - \ln|1-x| + \ln|1+x| + C$$

$$= \ln\left|\frac{1+x}{1-x}\right| - x + C$$

Arclength in Polar Coordinates

Let $r = f(\theta)$ be a polar function. Then

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

$$x' = f'\cos\theta - f\sin\theta$$
$$y' = f'\sin\theta + f\cos\theta$$

$$(x')^2 + (y')^2 = (f')^2\cos^2\theta - 2ff'\sin\theta\cos\theta + f^2\sin^2\theta$$
$$+ (f')^2\sin^2\theta + 2ff'\sin\theta\cos\theta + f^2\cos^2\theta$$
$$= f^2 + (f')^2$$

Thus $L = \int \sqrt{f^2 + (f')^2}\, d\theta$.

Polar coordinates simplify some arclength calculations.

Unit Circle $r = 1$

$$L = \int_0^{2\pi} \sqrt{1^2 + 0^2}\, d\theta = \theta\Big|_0^{2\pi} = 2\pi$$

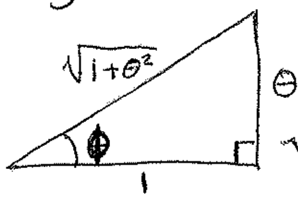
Logarithmic Spiral $r = e^\theta$

$$L = \int \sqrt{(e^\theta)^2 + (e^\theta)^2}\, d\theta = \int \sqrt{2}\, e^\theta\, d\theta = \sqrt{2}\, e^\theta + C$$

Spiral $r = \theta$

$$L = \int \sqrt{\theta^2 + 1^2}\, d\theta$$

Trig Substitution



$$\theta = \tan\phi$$
$$d\theta = \sec^2\phi\, d\phi$$
$$\sqrt{1+\theta^2} = \sec\phi$$

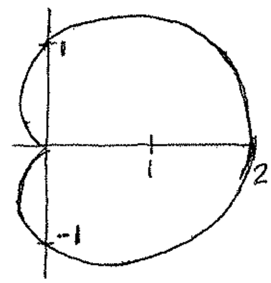
$$= \int \sec^3\phi\, d\phi$$

$$= \frac{1}{2}\sec\phi\tan\phi + \frac{1}{2}\ln\left|\sec\phi + \tan\phi\right| + C$$

$$= \frac{1}{2}\theta\sqrt{1+\theta^2} + \frac{1}{2}\ln\left|\sqrt{1+\theta^2} + \theta\right| + C$$

Limaçon (Cardioid)

$$r = 1 + \cos\theta, \quad \theta \in [0, 2\pi]$$



$$L = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{1 + \cos\theta}\, d\theta$$

Use the power-reducing formula

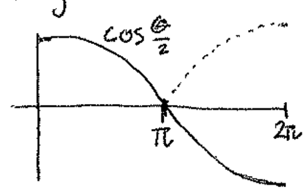
$$\cos^2 x = \frac{1 + \cos 2x}{2} \Rightarrow 2\cos^2\left(\frac{\theta}{2}\right) = 1 + \cos\theta$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{2\cos^2\left(\frac{\theta}{2}\right)}\, d\theta$$

$$= 2 \int_0^{2\pi} \left|\cos\left(\frac{\theta}{2}\right)\right| d\theta$$

Note that $\cos\frac{\theta}{2}$ is negative on $(\pi, 2\pi)$.

Use symmetry.



$$= 4 \int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta$$

$$= 8 \sin\frac{\theta}{2}\Big|_0^{\pi}$$

$$= 8$$