AREA BETWEEN CURVES

TOTAL AREA Suppose that a particle moves from point A to point B. The displacement of the particle between points A and B is the distance between the two points. The distance traveled by the particle is the length of the path it takes. These need not be the same.

Example. A particle moves right along the x-axis from $(0,0)$ to $(0,5)$. At this point, both the displacement and distance traveled between these two points are 5. It then moves left to $(0,3)$. The displacement between the origin and $(0,3)$ is 3, but the distance the particle traveled is 8.

Both quantities can be determined if we have a formula for the velocity function.

Example. A particle has velocity $v(t) = 2 - t$, $0 \le t \le 3$. Graphing the function, we note that it is nonnegative on $[0,2]$ and negative on $(2,3]$. The distance traveled between two times will be given by the area under the curve between those times. On $[0,2]$, the area under the curve is 2, while on $[2,3]$, the area is $\frac{1}{2}$ below the x-axis. The displacement is the difference of these two, $2 - \frac{1}{2} = \frac{3}{2}$. The distance traveled is the sum $2 + \frac{1}{2} = \frac{5}{2}$.

More generally, we can find areas under curves using an integral. Integration automatically treats areas below the curve as negative, so the displacement between times a and b is just $s(b) - s(a) = \int_a^b v(t)\,dt$, where $s(t)$ is the position function. To find the distance traveled, we need to add the areas below the x-axis rather than subtract them. The integral we wish to evaluate is $\int_a^b |v(t)|\,dt$.

We generalize this idea with the following definition.

Definition. The total area between $f(x)$ and the x-axis on $[a,b]$ is $\int_a^b |f(x)|\,dx$.

To evaluate the total area, we use the following method adapted from the previous example.

1. Subdivide $[a,b]$ where $f(x)$ changes sign (typically, at zeros).
2. Find the antiderivative $F(x)$.
3. Add absolute values of the integrals on the subintervals.

Example. Find the displacement and distance traveled by a particle with velocity $v(t) = 3t^2 - 18t + 24$ on $[0,5]$.

Solution. The position function $s(t) = t^3 - 9t^2 + 24t$ is an antiderivative, so the displacement is

$$\int_0^5 v(t)\,dt = \left(t^3 - 9t^2 + 24t\right)\Big|_0^5 = 20.$$

We have $v(t) = 3(t-2)(t-4)$, which has zeros of 2 and 4 and is negative only on $(2,4)$. Note that $s(0) = 0$, $s(2) = 20$, $s(4) = 16$, and $s(5) = 20$. Now the distance traveled is

$$\begin{aligned}
\int_0^5 |v(t)|\,dt &= \int_0^2 |v(t)|\,dt + \int_2^4 |v(t)|\,dt + \int_4^5 |v(t)|\,dt \\
&= \int_0^2 v(t)\,dt - \int_2^4 v(t)\,dt + \int_4^5 v(t)\,dt \\
&= [s(2) - s(0)] - [s(4) - s(2)] + [s(5) - s(4)] \\
&= -s(0) + 2s(2) - 2s(4) + s(5) \\
&= 28
\end{aligned}$$

Notice the pattern of coefficients $-1, 2, -2, 1$ in the next-to-last line. This can be generalized as follows.

Theorem. Let $f(x)$ change signs at x_1, ..., x_n, where $a < x_1 < ... < x_n < b$. Then

$$\int_a^b |f(x)|\, dx = -F(a) + 2F(x_1) - 2F(x_2) + 2F(x_3) - ... \pm 2F(x_n) \mp F(b), \quad f > 0\, on\ (a, x_1).$$

$$\int_a^b |f(x)|\, dx = \quad F(a) - 2F(x_1) + 2F(x_2) - 2F(x_3) + ... \pm 2F(x_n) \mp F(b), \quad f < 0\, on\ (a, x_1).$$

Example. Find the total area between $f(x) = x^{\frac{1}{3}} - x$ and the x-axis on $[-1, 8]$.

Solution. We observe that 0, 1, and -1 are zeros of $f(x)$. This helps find the factorization $f(x) = x^{\frac{1}{3}}\left(1 - x^{\frac{2}{3}}\right) = x^{\frac{1}{3}}\left(1 - x^{\frac{1}{3}}\right)\left(1 + x^{\frac{1}{3}}\right)$. The antiderivative is $F(x) = \frac{3}{4}x^{\frac{4}{3}} - \frac{1}{2}x^2$. We note that f is nonnegative on $[0, 1]$ and negative elsewhere. Thus the total area is $\int_{-1}^{8} |f(x)|\, dx = F(-1) - 2F(0) + 2F(1) - F(8) = \frac{1}{4} + 2\frac{1}{4} - (-20) = \frac{83}{4}$.

AREA BETWEEN CURVES What if we want to find the area between two functions $f(x)$ and $g(x)$? If $f(x) \geq g(x)$, we would expect the area A to satisfy $\int_a^b g(x)\, dx + A = \int_a^b f(x)\, dx$. This leads to the following definition.

Definition. If $f(x) \geq g(x)$ on $[a, b]$, the area between $f(x)$ and $g(x)$ on $[a, b]$ is $\int_a^b (f(x) - g(x))\, dx$. If $f(x)$ and $g(x)$ cross, the area between them is the total area $\int_a^b |f(x) - g(x)|\, dx$.

Thus we see that the total area and area between curves are essentially the same concept, as the total area of $f(x)$ is the area between $y = f(x)$ and $y = 0$, while the area between $f(x)$ and $g(x)$ is the total area of $f(x) - g(x)$.

Example. Find the area enclosed by $y = x^4$ and $y = 8x$.

Solution. Graphing the functions, it appears that the functions have two intersections. To find the intersections, set them equal and solve for x. We see $x^4 = 8x$, so $x^4 - 8x = 0$, so $x(x^3 - 8) = 0$, so $x = 0$ or $x = 2$. Since $8x \geq x^4$ on $[0, 2]$, the area is $A = \int_0^2 (8x - x^4)\, dx = \left(4x^2 - \frac{1}{5}x^5\right)\Big|_0^2 = \frac{48}{5}$.

Example. Find the area enclosed by $y = \sin x$ and $y = \cos x$ on $[0, 2\pi]$.

Solution. The functions intersect when $\sin x = \cos x$. Then $\tan x = 1$, so $x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$. Now the antiderivative of $f(x) = \cos x - \sin x$ is $F(x) = \sin x + \cos x$. Now $A = \int_0^{2\pi} |\cos x - \sin x|\, dx = -F(0) + 2F\left(\frac{\pi}{4}\right) - 2F\left(\frac{5\pi}{4}\right) + F(2\pi) = -1 + 2\sqrt{2} - 2\left(-\sqrt{2}\right) + 1 = 4\sqrt{2}$.

Note that if we had failed to remember the intersections, we would have incorrectly concluded that the area is $\int_0^{2\pi} (\cos x - \sin x)\, dx = 0$.

Sometimes it is more convenient to integrate with respect to y.

Example. Find the area between $y = \sqrt{x}$, $y = x - 2$, and the x-axis.

Solution. Graphing the curves shows that there are two different functions that make the bottom of the region. We could split the region into two and evaluate two separate integrals. But if we integrate with respect to y, we can do only one integral. The curves $y = \sqrt{x}$, $y = x - 2$ intersect when $y = y^2 - 2$, so $(y - 2)(y + 1) = 0$. Only $y = 2$ yields a valid solution. Thus the area is $A = \int_0^2 (y + 2 - y^2)\, dy = \left(\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right)\Big|_0^2 = \frac{10}{3}$.