Collapsible Graphs

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Abstract

A graph G is k-collapsible if $\delta(G) = k$ and any proper induced subgraph has smaller minimum degree. Collapsible graphs can be viewed as lower extremal graphs for monocore graphs. We investigate the structure and basic properties of these graphs. A k-collapsible graph necessarily has many vertices of degree k; we seek sharp bounds on how many it must have.

1 Introduction

Definition 1. The k**-core** of a graph G, $C_k(G)$, is the maximal induced subgraph $H \subseteq G$ such that the minimum degree $\delta(H) \geq k$, if it exists.

The k-core can be found by iteratively deleting vertices of degree less than k until none remain (the k**-core algorithm**).

Definition 2. A graph is k**-degenerate** if its vertices can be successively deleted so that when deleted, each has degree at most k. The **degeneracy** $D(G)$ of a graph G is the smallest k such that it is k-degenerate.

There are natural bounds on degeneracy: $\delta(G) \leq D(G) \leq \Delta(G)$.

Definition 3. A **monocore graph** G is a graph with $\delta(G) = D(G)$.

This author began the study of monocore graphs in [1]. Many important graph classes are monocore (regular graphs, trees, wheels, complete multipartite, maximal outerplanar). To understand these graphs, we can examine their extremal graphs.

One common technique for understanding a class of graphs is examining its extremal graphs. That is, graphs in the class which either cannot have any edges added or cannot have any edges deleted without ceasing to be in that class. We will call these two types of extremal graphs maximal and minimal extremal graphs, respectively.

We will first examine the maximal extremal k-monocore graphs. In fact, these are just maximal k-degenerate graphs. Maximal k-degenerate graphs are clearly k-monocore. A partial converse to this fact is true.

Lemma 4. [3] *Every k-monocore graph is contained in a maximal k-degenerate graph.*

Maximal k-degenerate graphs have been well-studied [5, 6, 4, 2]. This author characterized the size and degree sequences of monocore graphs in [3].

Proposition 5. [3] *The size m of a k-monocore graph G of order n satisfies*

$$\left\lceil \frac{k \cdot n}{2} \right\rceil \leq m \leq k \cdot n - \binom{k+1}{2}.$$

Theorem 6. [3] *A nonincreasing sequence of integers $d_1, \ldots, d_n$ is the degree sequence of some k-monocore graph G, $0 \leq k \leq n-1$, if and only if $k \leq d_i \leq min\{n-1, k+n-i\}$ and $\sum d_i = 2m$, where m satisfies $\lceil \frac{k \cdot n}{2} \rceil \leq m \leq k \cdot n - \binom{k+1}{2}$.*

2 k-Collapsible Graphs

Next we consider a variation on the lower extremal k-monocore graphs, where instead of considering deleting an edge, we consider deleting vertices.

Definition 7. A graph G is **k-collapsible** if it is k-monocore and has no proper induced k-core.

This immediately implies that a k-monocore graph is k-collapsible if and only if for every vertex v in G, $G - v$ has no k-core.

For small values of k, the following characterizations of k-collapsible graphs are immediate.

Proposition 8. *Let G be a graph.*
G is 0-collapsible if and only if $G = K_1$.
G is 1-collapsible if and only if $G = K_2$.
G is 2-collapsible if and only if G is a cycle.

The structure of k-collapsible graphs is considerably more complicated for $k > 2$.

Collapsible graphs are interesting in part because every k-monocore graph contains one.

Proposition 9. *Every k-monocore graph G contains a k-collapsible graph as an induced subgraph. Indeed, every component of G contains such a subgraph.*

Proof. If G has order $n = k + 1$, then the unique k-monocore graph of that order, $G = K_{k+1}$ is k-collapsible. Assume the result holds for all k-monocore graphs with order at most r, and let G have order $r+1$. If $G-v$ has no k-core for all v in G, then G is k-collapsible. If not, then there is some vertex v in G so that $G-v$ has a k-core. Let H be the k-core of $G-v$. Then H is an induced subgraph of G with order at most r, so by induction it contains a k-collapsible subgraph. The final statement holds since every component of a k-monocore graph is k-monocore. □

We can offer a characterization of sorts for k-collapsible graphs.

Definition 10. A **barrier** in a k-monocore graph is a minimal cutset $S \subset V(G)$ such that for some component H of $G - S$, every vertex v of S has $d_{G-H}(v) \geq k$.

Note that every vertex in a barrier of a k-monocore graph G necessarily has degree greater than k in G.

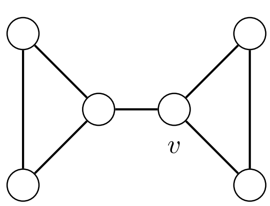
Proposition 11. *A k-monocore graph G is collapsible if and only if it does not have a barrier.*

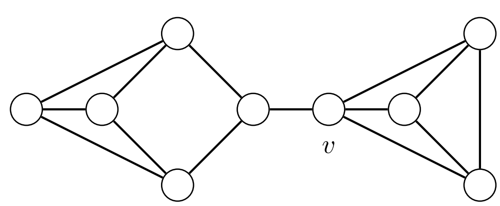
Proof. If G has a barrier S and corresponding component H, then the vertices of $G - H$ all have degree at least k. Thus G has a proper k-core, so it is not collapsible.

If G is not collapsible, then it has a proper induced subgraph F such that is a k-core. Then the vertices of F adjacent to vertices of $G-F$ must be a barrier. □

Checking every set of vertices, or even every cutset of vertices of degree greater than k is not efficient. It is easier to determine whether G has a barrier by running the k-core algorithm on $G - v$ for all v.

A barrier need not be a large set or have large degrees. Indeed, for all $k \geq 3$, there is a k-monocore graph G with a barrier of one vertex of degree $k+1$ or $k+2$. If k is odd, a barrier need only have one vertex of degree $k+1$, while if k is even, it need have no more than two. In the graphs below, $\{v\}$ is a barrier. These constructions generalize for $k > 3$.





Several results on collapsible graphs follow immediately.

Corollary 12. *A regular graph is collapsible if and only if it is connected.*

Let G be a graph with $\delta(G) \geq k$ and $v \in V(G)$. Then $C_k(G - v) = C_k(G) - v$ if and only if every neighbor of v has degree at least $k+1$.

If G is k-collapsible, then every vertex of G is adjacent to a vertex of degree k. This implies that the vertices of degree k of a k-collapsible graph G form a total dominating set of G.

It is not the case that if the vertices of degree k form a total dominating set, the graph is collapsible. This can be seen, for example, in the graphs constructed above. Also, the graph formed by adding a perfect matching between the vertices of C_n and K_n, $n > 3$, is not 3-collapsible, even though the total dominating set is connected.

Theorem 13. *For $k \geq 1$, the size of a k-collapsible graph G of order n satisfies*

$$\left\lceil \frac{k \cdot n}{2} \right\rceil \leq m(G) \leq (k-1) \cdot n - \binom{k}{2} + 1.$$

Proof. The lower bound follows since the sum of degrees is at least $k \cdot n$.

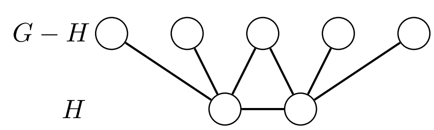
For the upper bound, let G be k-collapsible and e an edge of G incident with a vertex v of degree k. Then $G - e$ is $k-1$-degenerate, since $d_{G-e}(v) < k$ and G has no proper induced k-core. Then $G - e$ is contained in a maximal $k-1$-degenerate graph H, and $G \subseteq H + e$. Thus $m \leq (k-1) \cdot n - \binom{k}{2} + 1$. $\square$

The upper bound of this theorem is sharp. For example, for $k \geq 3$ the graph $G = C_{n-k+2} + K_{k-2}$ achieves the upper bound and is k-collapsible since every vertex on the cycle has degree k and G has no k-core not containing one of them. This example also shows that for all k, n with $3 \leq k \leq n-1$ there is a k-collapsible graph with maximum degree $n-1$. Along with the cycle C_n, this shows that for all k, n with $2 \leq k \leq n-1$ there is a k-collapsible graph of order n.

3 Degrees of k-Collapsible Graphs

It is natural to ask how many vertices of degree k a k-collapsible graph must contain.

Theorem 14. *For $k \geq 3$, every k-collapsible graph G of order n has at least $\left\lceil \frac{2}{2k-1} n \right\rceil$ vertices of degree k, and hence at most $\left\lfloor \frac{2k-3}{2k-1} n \right\rfloor$ vertices of degree more than k.*



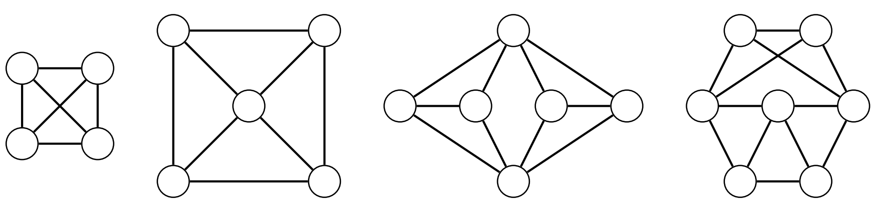
Proof. Let G be k-collapsible, $k \geq 3$. Then each vertex (including those of degree k) is adjacent to a vertex of degree k. Consider a component H of the graph induced by the vertices of degree k, which has order $r \geq 2$. Since it is connected, it has size at least $r-1$, so there are at most $kr - 2(r-1)$ edges between H and $G - H$. At least two of these edges must be incident with the same vertex in $G-H$ for G to collapse, so H has at most $kr-2r+1$ neighbors in $G - H$. Now $\frac{kr-2r+1}{r}$ is maximized on $r \geq 2$ when $r = 2$, in which case the maximum is $k - \frac{3}{2}$. Thus if there are s vertices of degree k in G, there are at most $\left(k - \frac{3}{2}\right) s$ vertices of larger degree. Thus G has at least $\frac{s}{\left(k-\frac{3}{2}\right)s+s} = \frac{1}{k-\frac{1}{2}} = \frac{2}{2k-1}$ of its vertices with degree k. □

Lemma 15. *A k-collapsible graph of order $n \geq k+1$ contains at least $k^2 - k - 2 - (k-3)\, n$ vertices of degree k.*

Proof. The size of a k-collapsible graph is at most $(k-1) \cdot n - \binom{k}{2} + 1$, so the sum of degrees is at most $2\left[(k-1) \cdot n - \binom{k}{2} + 1\right]$. Since the minimum degree is k, there are at most $2\left[(k-1) \cdot n - \binom{k}{2} + 1\right] - kn = (k-2)\, n - k^2 + k + 2$ vertices of degree more than k, and at least $n - \big((k-2)\, n - k^2 + k + 2\big) = k^2 - k - 2 - (3-k)\, n$ of degree k. □

This raises the question of when these bounds are attained.

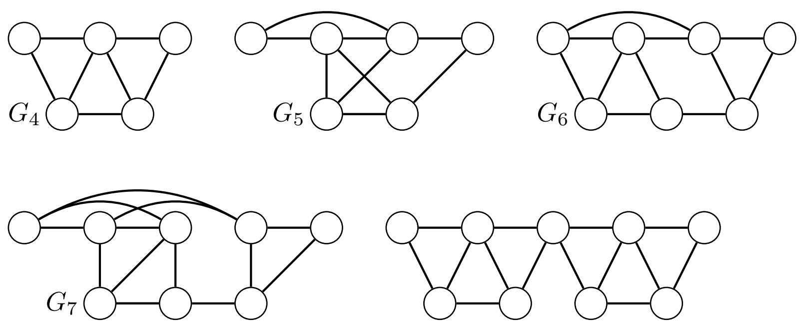
Theorem 16. *The minimum number of vertices of degree 3 in a 3-collapsible graph of order $n \geq 4$ is* $\max\left\{4, \left\lceil \frac{2}{5} n \right\rceil\right\}$.



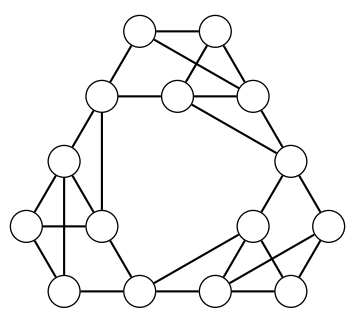
Proof. For 3-collapsible graphs, Lemma 15 gives a bound of 4. For orders $4 \leq n \leq 7$, this is achieved for the graphs above. Consider the graphs $G_4 = P_4 + K_1$, G_5, G_6, and G_7 defined in the figure below.

We construct larger graphs by arranging some of these graphs together in a cycle and identifying vertices of degree two (or 2 and 3) in consecutive graphs. For order 8, we use two copies of G_4. For order 9, we use a G_4 and a G_5. In general, for $n \geq 8$, we use $\left\lceil \frac{n-8}{5} \right\rceil$ copies of G_5 and either one G_4, G_5, G_6, G_7 or two G_4s to add the remaining vertices. These graphs have $\left\lceil \frac{2}{5} n \right\rceil$ degree 3 vertices.

To see that these graphs are all 3-collapsible, we note that each vertex of degree more than 3 is adjacent to one of degree 3. Each vertex of degree 3 is adjacent to another of degree 3. Each connected set of vertices of degree 3 has a common neighbor of larger degree, so all vertices in their G_i collapse. This causes the collapse of a vertex in G_{i+1}, and eventually the entire graph. $\square$



A 3-collapsible graph of order 15 with only six vertices of degree 3 is displayed below.

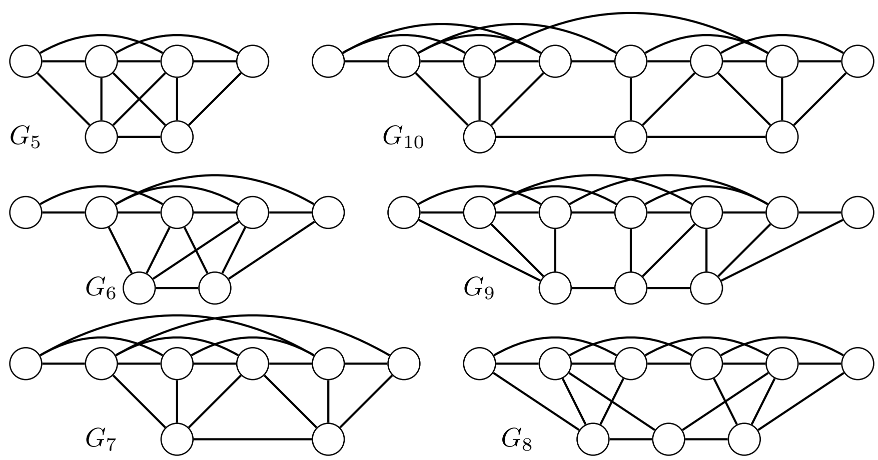


Theorem 17. *The minimum number of vertices of degree 4 in a 4-collapsible graph of order* $n \geq 5$ *is* $\max\left\{10-n, \left\lceil \frac{2}{7}n \right\rceil\right\}$.

Proof. For 4-collapsible graphs, Lemma 15 gives a bound of $10-n$. For orders $5 \leq n \leq 7$, this is achieved for K_5, $K_6 - 2K_2$, and $\overline{P_3 \cup P_4}$. Consider the graphs $G_5 = P_4 + K_2$, G_6, G_7, G_8, G_9, and G_{10} defined in the figure below.

We construct larger graphs by arranging some of these graphs together in a cycle and identifying vertices of degree two or three in consecutive graphs. For orders 8, 9 and 10, we use G_8, G_9 and G_{10}, in each case identifying the two vertices of degree 3. For $n \geq 11$, we use $\left\lceil \frac{n-11}{7} \right\rceil$ copies of G_7 and either one G_5, G_6, G_7, G_8, G_9, G_{10} or a G_5 and G_6 to add the remaining vertices. Each of these graphs has $\left\lceil \frac{2}{7}n \right\rceil$ vertices of degree 4.

To see that these graphs are all 4-collapsible, we note that each vertex of degree more than 4 is adjacent to one of degree 4. Each vertex of degree 4 is adjacent to another of degree 4. Each connected set of vertices of degree 4 has a common neighbor of larger degree, so all vertices in their G_i collapse. This causes the collapse of a vertex in G_{i+1}, and eventually the entire graph. $\square$



Conjecture 18. *The minimum number of vertices of degree k in a k-collapsible graph of order $n \geq k+1$ is* $\max\left\{k^2 - k - 2 - (k-3)\,n, \left\lceil \frac{2}{2k-1} n \right\rceil\right\}$.

For $k = 5$, this claims that the minimum number of vertices of degree 5 in a 5-collapsible graph of order $n \geq 6$ is $\max\left\{18 - 2n, \left\lceil \frac{2}{9} n \right\rceil\right\}$. For larger values of k , we can show that the bound of Theorem 14 is sharp.

Theorem 19. *The bound in Theorem 14 is sharp. In particular, for every $k \geq 5$ and order $n = (2k-1)\,i$, $i \in \mathbb{Z}^+$, there is a k-collapsible graph that has $\left\lfloor \frac{2k-3}{2k-1} n \right\rfloor$ vertices of degree more than k.*

Proof. We will show that when k is odd, there is a graph G of order $2k$ with two adjacent vertices of degree k with one common neighbor and $2k-3$ total neighbors, of which $2k-2$ have degree $k+1$. Also, G has two nonadjacent vertices of degree $\frac{k+1}{2}$ with no common neighbors. If such a graph exists, the two vertices of degree $\frac{k+1}{2}$ can be identified, forming one vertex of degree $k+1$. If the two vertices with degree k are then deleted, the resulting graph H has degree sequence $(k)_{2k-4}\,(k-1)_1$. Using the characterization of degree sequences of monocore graphs (Theorem 6), it is easily checked that this sequence satisfies the conditions for that of a $k-1$-monocore graph. Thus H exists, so G does also.

If k is even, we want a graph of order $2k$ with two adjacent vertices of degree k with one common neighbor and $2k-3$ total neighbors, of which $2k-2$ have degree $k+1$. It must also have two nonadjacent vertices of degree $\frac{k+2}{2}$ with no common neighbors. If such a graph exists, the two vertices of degree $\frac{k+2}{2}$ can be identified, forming one vertex of degree $k+2$. If the two vertices with degree k are then deleted, the resulting graph has degree sequence $(k+1)_1\,(k)_{2k-3}\,(k-1)_1$. Using Theorem 6, it is easily checked that this sequence satisfies the conditions for that of a $k-1$-monocore graph. Thus H exists, so G does also.

In either case, we use i copies of G and identify consecutive vertices of degree $\frac{k+1}{2}$ or $\frac{k+2}{2}$ to form a graph of order $(2k-1)\,i$. In this graph, $\left\lceil \frac{2}{2k-1}n \right\rceil$ vertices have degree k, and $\left\lfloor \frac{2k-3}{2k-1}n \right\rfloor$ vertices have degree more than k. Each vertex of degree more than k is adjacent to one of degree k, each vertex of degree k is adjacent to another one, and each pair of adjacent vertices of degree k have a common neighbor. Without this pair, the copy of G containing them collapses, and so does the entire graph. Hence it is k-collapsible. $\square$

It is likely possible to fill in the other orders for all larger k, but no simple construction to achieve this is apparent.

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