COUNTING TECHNIQUES AND IDENTITIES

1 Ordered Counting

In combinatorics, we often wish to count the number of objects in a set. In graph theory, we may wish to find the number of some type of graph or the number of some type of subgraph of a graph. We begin with a basic fact.

Theorem 1. *The Counting Product Rule*

If there are k independent choices, which can be made in n_i ways, $1 \leq i \leq k$, then there are $\prod n_i$ total ways to make the choices.

Example. A Michigan license plate has three letters followed by four numbers. There are $26^3 10^4 = 175,760,000$ possible license plates.

Example. How many positive integer factors does 360 have?

Solution. The unique prime factorization of 360 is $2^3 3^2 5$. There are four possibilities (0, 1, 2, or 3) for how many 2's can be in a factor. Similarly, there are three possibilities for the 3's, and two for the five. Thus there are $4 \cdot 3 \cdot 2 = 24$ total factors. Note that even if we listed all 24, we could not be sure we had them all without a counting argument.

Since any positive integer has a unique prime factorization $n = p_1^{n_1} \cdots p_k^{n_k}$, the number of positive integer factors $\tau(n)$ of n is $\tau(n) = (n_1 + 1) \cdots (n_k + 1)$.

Example. How many subsets does a set with n elements have?

Solution. For each element, there are two possibilities. It is either in or out of a subset. This is true for each element, so there are 2^n subsets.

Example. There are n^{n-2} sequences of length $n-2$ with elements from $[n] = \{1, ..., n\}$. This is also the number of labeled trees, as there is a bijection between such sequences and labeled trees.

Suppose that we want to know how many ways there are to select k objects out of n possible objects. There are two questions we need to ask.

- Are repetitions allowed?
- Is order important?

If repetitions are allowed and order is important, then there are n possibilities for the first choice, n for the second choice, etc. Thus there are $n^k = n \cdot ... \cdot n$ ways to select the k objects.

Suppose repetitions are not allowed and order is important, and consider the special case $n = k$. Such an arrangement of n objects is called a permutation. There are n possibilities for the first choice. There are $n - 1$ for the second choice since one object has already been picked and is no longer available. Continue similarly down to 1 possibility for the last choice. Thus there are a total of $n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n!$ ways to order the objects.

Corollary 2. *There are $\frac{n!}{(n-k)!}$ ways to list k of n distinct objects. In particular, there are $n!$ permutations of n objects.*

Proof. There are $n(n-1)\cdots(n-k+1) = n(n-1)\cdots(n-k+1)\frac{(n-k)\cdots 3\cdot 2\cdot 1}{(n-k)\cdots 3\cdot 2\cdot 1} = \frac{n!}{(n-k)!}$ possible lists.

□

Example. In a race with ten horses, there are $10 \cdot 9 \cdot 8 = 720$ possibilities for win, place, and show.

Factorials can be defined recursively as $0! = 1$ and $n! = n \cdot (n-1)!$ for any positive integer n. Computing the first few values, we have the following.

n	1	2	3	4	5	6	7	8
$n!$	1	2	6	24	120	720	5040	40320

It is apparent that factorials grow very quickly. Stirling's approximation for $n!$ is $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. This can be used to show that $n!$ grows faster than any exponential function, but slower than n^n.

2 Unordered Counting

What if there are repeated elements in a permutation? Suppose a of n elements are the same. If we treat each as distinct, there are $n!$ permutations. But the a elements are permuted in $a!$ ways, so each ordering occurs $a!$ times. Thus we must divide by $a!$, so there are $\frac{n!}{a!}$ distinct permutations. We can generalize this.

Theorem 3. *Suppose n objects with k distinct types repeated a_i times, $1 \leq i \leq k$, with $a_1 + ... + a_k = n$ are ordered. There are $\frac{n!}{a_1! \cdots a_k!}$ ways to order them.*

Example. A multiset is a collection of objects where repetition is allowed. Consider the multiset $\{A, A, A, B, B, C\}$. There are $\frac{6!}{3!2!1!} = 60$ orders for this set.

How many ways can k objects be chosen from a set of n objects if repetition is not allowed order is unimportant? We call this the number of combinations "n choose k", with notation $\binom{n}{k}$. We have k chosen and $n-k$ unchosen elements, which justifies the following.

Corollary 4. *The number of ways k objects can be chosen from a set of n objects if repetition is not allowed order is unimportant is*

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n\,(n-1) \cdot ... \cdot (n-k+1)}{k!}.$$

Example. In a class of seven students, there are $\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ possible groups of three students.

Example. How many labeled graphs with order n are there?

Solution. There are $\binom{n}{2}$ possible edges. Each edge can be included or not in a graph. Thus there are $2^{\binom{n}{2}}$ labeled graphs of order n.

Example. How many 3-cubes are contained in the n-cube?

Solution. Think of the vertices as bit strings. Any 3-cube must vary three of the bits. There are $\binom{n}{3}$ ways to do this. There are 2^{n-3} choices for the other bits. Thus there are $\binom{n}{3}2^{n-3}$ 3-cubes in the n-cube.

How many ways can k objects be chosen from a set of n objects if repetition is allowed and order is unimportant? Every choice of k objects from a multiset can be thought of as putting x's in bins corresponding to the elements. Equivalently, we could find the number of orderings of k x's and $n-1$ dividers. This justifies the following result.

Corollary 5. *The total number of unordered choices of k elements from a set of n elements, allowing repetition, is $\binom{k+n-1}{k}$.*

Suppose we want to find how many ways are there to write a positive integer N as a sum of p whole numbers. Let the p whole numbers be bins and distribute the N 1's. Then there are $\binom{N+p-1}{N}$ possible sums.

Example. If $N = 4$ and $p = 3$, we have $\binom{4+3-1}{4} = \binom{6}{4} = 15$ sums.

The formulas for the number of ways to select k of n objects are summarized in the following table.

	Repetitions Allowed	No Repetitions
Order Important	n^k	$\frac{n!}{(n-k)!}$
Order Unimportant	$\binom{n+k-1}{k-1}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

3 Combinatorial Identities

Consider the expression $(x+y)^n$ (n an integer). We can use the distributive property to completely expand it out. Each term of the resulting sum has one factor being either x or y from each of the n binomials. For example,

$$(x+y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy = x^3 + 3x^2y + 3xy^2 + y^3.$$

Thus there will be 2^n terms, some of which will be equivalent. The term $x^k y^{n-k}$ will occur every time exactly k of the x's are chosen from the n binomials. Thus it will occur $\binom{n}{k}$ times. This justifies the following theorem.

Theorem 6. *[Binomial Theorem] Let n be a nonnegative integer. Then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

This is the Binomial Theorem. It can also be proved by induction. The Binomial Theorem has several immediate consequences.

Corollary 7. *Let n be a nonnegative integer. Then $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.*

Proof. Plug $x = y = 1$ into the Binomial Theorem.

□

Corollary 8. *Let n be a nonnegative integer. Then $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + ... = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + ... = 2^{n-1}$.*

Proof. Plug $x = -1$ and $y = 1$ into the Binomial Theorem. Then

$$0 = (-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - ...$$

so $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + ... = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + ...$ Now both sides together sum to 2^n, so each individually sums to 2^{n-1}.

□

The numbers $\binom{n}{k}$ are known as the binomial coefficients. They can be arranged as follows, where the rows give increasing values of n, and the diagonals give increasing values of k.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

This is called Pascal's Triangle. Corollary 7 can be interpreted as finding the sums of the rows of Pascal's Triangle. There are many other identities related to Pascal's Triangle and binomial coefficients. We introduce two techniques for generating and proving combinatorial identities.

Definition 9. A partition of a set S is a collection of nonempty, mutually disjoint subsets S_i of S whose union is S. The Counting Sum Rule says that for any partition of S, the cardinality of S is the sum of the cardinalities of the subsets in the partition,$|S| = \sum |S_i|$.

Counting Two Ways. The first new technique is partitioning a set in two different ways, and using each partition to find an expression for the cardinality of the set. Since both expressions count the same cardinality, they must be equal, proving a combinatorial identity. This technique is known as Counting Two Ways.

Pascal's Triangle can be simply generated by noting that $\binom{n}{0} = \binom{n}{n} = 1$, and each number in its interior is the sum of the two numbers diagonally above it. For example, $10 = 4 + 6$. This works because of the following identity.

Proposition 10. *[Pascal's Formula] For any integers $n \geq 1$ and k, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.*

Proof. Consider the set of all k-element subsets of $[n]$. There are $\binom{n}{k}$ of them. Partition the set into the subsets containing 1 and not containing 1. There are $\binom{n-1}{k-1}$ subsets containing 1 and $\binom{n-1}{k}$ subsets not containing 1. Summing them proves the identity.

□

This identity, and all the others in this section can also be proved algebraically. However, there is good reason to prefer the combinatorial approach. A combinatorial argument gives a natural explanation of the result that algebra does not. Counting arguments are also a natural technique for generating new combinatorial identities. Algebra can prove existing identities, but it does not naturally suggest new identities.

The row sum identity (Corollary 7) proved above also has a natural combinatorial interpretation and proof.

Proposition 11. *Let n be a nonnegative integer. Then $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.*

Proof. Consider the set of all subsets (the power set) of $[n]$, which has cardinality 2^n. Partition this set based on the cardinalities of the subsets. There are $\binom{n}{k}$ subsets of cardinality k, so summing from 0 to n counts all subsets.

□

Another identity comes from summing a diagonal of Pascal's Triangle up to a point.

Proposition 12. *We have $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + ... + \binom{n}{k} = \binom{n+1}{k+1}$.*

Proof. Consider the set of all $k+1$-element subsets of $[n+1]$, which has cardinality $\binom{n+1}{k+1}$. The largest number chosen must be between $k+1$ and $n+1$. Partition the set based on the largest number. If it is $k+i+1$, there are $\binom{k+i}{k}$ ways to choose the other k elements. Summing between k and n counts all the subsets.

□

If $k = 1$, this identity reduces to $1 + 2 + 3 + ... + n = \binom{n}{2}$. This identity is a common example used to teach proof by mathematical induction. While this is a valid method of proof, it does not explain where the identity comes from, or what it means, as does the counting argument.

Counting two ways is used to prove the First Theorem of Graph Theory.

Theorem 13. *For a graph G, we have $\sum d(v_i) = 2m$.*

Proof. Consider the set of all vertex-edge incidences (the "ends" of edges) in a graph. Partitioning the set by vertices shows its cardinality is the sum of the degrees. Partitioning the set by edges shows each edge appears twice, so its cardinality is $2m$. Thus $\sum d(v_i) = 2m$.

□

Counting by Bijection. In this counting technique, we find a bijection between the elements of two distinct finite sets. This shows that they have the same cardinality. If we can count both sets, we establish a combinatorial identity. If we can count only one of the sets, we find a formula for the other set's cardinality.

This technique is different from counting two ways. In that technique, we partition one set in two different ways; in counting by bijection, we find a bijection between two (usually) different sets. In the first technique, the problem is finding the partitions, in the second the problem is finding the bijection and then counting one or both sets.

Proposition 14. *Pascal's triangle is symmetric: $\binom{n}{k} = \binom{n}{n-k}$.*

Proof. Establish a bijection between the set of k-element subsets of $[n]$ and the set of $n-k$-element subsets of $[n]$ where each set is mapped to its complement in $[n]$. The first set has $\binom{n}{k}$ elements, and second has $\binom{n}{n-k}$ elements. The bijection shows these quantities are equal.

□

The alternating sum identity for Pascal's triangle also has a combinatorial proof.

Proposition 15. *We have $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + ... = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + ...$*

Proof. Consider the set of subsets of $[n]$ with even cardinality and the set of subsets of $[n]$ with odd cardinality. They have cardinalities $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + ...$ and $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + ...$, respectively. Establish a bijection between them by deleting n from a subset if it appears and adding it if it does not. This function always changes the cardinality by one and it is its own inverse, so it is a bijection.

□

Counting by bijection is a common tool in graph theory. A bijection between the set of labeled trees and the set of Prufer sequences is used to count labeled trees. When some class of graphs has large sizes, it may be easier to consider the bijection to their complements and count them instead. Bijections can also be used to count subgraphs within a graph.

Proposition 16. *The Petersen graph contains 15 8-cycles.*

Proof. Any 8-cycle omits two vertices. They must be adjacent, or else they would have a common neighbor that could not be on the cycle. Deleting two adjacent vertices from the Petersen graph results in a graph that clearly contains a single 8-cycle. The Petersen graph has 15 edges, and hence 15 pairs of adjacent vertices, so it has 15 8-cycles.

□

Many graph theory counting problems appear throughout the exercises of this text.

Related Terms: Inclusion-Exclusion Formula, derangement, partition number, Bell number, Catalan number, Stirling number, generating function, Euler's phi function

Exercises.

1. Simplify $\frac{(n+1)!}{n!}$.
2. Which is bigger, $(2n)!$ or $2 \cdot n!$?
3. Is there a natural choice for the value of $(-1)!$? If so, what is it?
4. Express the product $(2n-1)(2n-3)\cdots 5 \cdot 3 \cdot 1$ using factorials.
5. How many different factors do the following numbers have?
a. 105 b. $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ c. 2^{10} d. 120
6. How many different factors do the following numbers have?
a. 48 b. 1729 c. 2^k d. 10^k
7. How many 7-digit numbers have
a. No 0
b. No repeated digits
c. No consecutive repeated digits
8. A math class has 7 students.
a. How many different orders are there for three students to make class presentations?
b. How many different groups of three students could make a presentation?
9. How many n-digit positive integers are there?

10. A combination lock requires three distinct numbers out of 20 be entered. If you forgot the combination, how many possible combinations are there to try?

11. Out of a group of 10 students, how many different groups are there that could go on a field trip?

12. Out of eight runners, how many possibilities are there for the winners of the gold, silver, and bronze medals?
13. How many paths of length $2n$ are there from the point $(0,0)$ to (n,n) in a grid?
14. The Catalan numbers are a sequence that occurs in many counting problems with formula $C_n = \frac{1}{n+1}\binom{2n}{n}$.
a. Calculate C_n for $0 \leq n \leq 5$.
b. Use Stirling's approximation to find an approximation for C_n.
15. How many ways are there to write 5 as the sum of three whole numbers?
16. How many ways are there to write 8 as the sum of four whole numbers?
17. Expand the binomial $(x+2y)^3$.
18. Expand the binomial $(2x+3y)^5$.

19. Generate the next two rows of Pascal's triangle.

20. Color the odd and even numbers in Pascal's triangle differently. What pattern do you find?

21. Verify $\binom{n}{k} = \binom{n}{n-k}$ algebraically.
22. Verify $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ algebraically.

23. Find the sums of the first ten '$\frac{1}{3}$ slope' diagonals of Pascal's triangle: 1, 1, 1+1, 1+2, 1+3+1, ... What familiar pattern do you find? Explain why this works.

24. Show that the rows of Pascal's triangle increase from $k=0$ to $k = \lfloor \frac{n+1}{2} \rfloor$ and decrease from $k = \lceil \frac{n+1}{2} \rceil$ to $k=n$.

25. Partition the power set of $[n]$ according to the largest element in each subset. What combinatorial identity follows from this?
26. Consider mapping each subset of $[n]$ to its complement. Can this be used to prove Proposition 15?
27. Show that $k\binom{n}{k} = n\binom{n-1}{k-1}$ using
a. counting two ways (hint: consider choosing committees with a chairman)
b. algebra
28. Show that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ using
a. counting two ways (hint: consider choosing committees with a chairman)
b. calculus and the Binomial Theorem
29. (VandeMonde's Identity) Show that $\binom{n+m}{k} = \sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}$ using counting two ways.
30. Show that $\binom{n}{m}\binom{n-m}{k} = \binom{n}{k}\binom{n-k}{m}$, $k+m \leq n$, using counting two ways.
31. Show that $\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2$.
32. + Use induction on n to prove the Binomial Theorem.