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NOTE

# MINIMUM EDGE CUTS IN DIAMETER 2 GRAPHS

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**Abstract**

Plesnik proved that the edge connectivity and minimum degree are equal for diameter 2 graphs. We provide a streamlined proof of this fact and characterize the diameter 2 graphs with a nontrivial minimum edge cut.

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Let $G$ be a graph. For $S, T \subseteq V(G)$, let $[S,T]$ be the set of edges with one end in $S$ and the other in $T$. An **edge cut** of a graph $G$ is a set $X = [S,T]$, of edges so that $G - X$ has more components than $G$. The **edge connectivity** $\lambda(G)$ of a connected graph is the smallest size of an edge cut. A disconnected graph has $\lambda(G) = 0$. Often we can express an edge cut as $\left[S, \overline{S}\right]$, where $\overline{S} = V(G) - S$.

Denote the **minimum degree** of $G$ by $\delta(G)$. It is well-known that $\lambda(G) \leq \delta(G)$, since the edges incident with a vertex of minimum degree form an edge cut. Plesnik proved that this is an equality for diameter 2 graphs. We present a shorter proof.

**Theorem 1.** [3] *If $G$ has diameter 2, then $\lambda(G) = \delta(G)$.*

***Proof.*** Let $[S, \overline{S}]$ be a minimum edge cut. Now $S$ and $\overline{S}$ cannot both have vertices $u$ and $v$ that are not incident with $[S, \overline{S}]$, for then $diam(G) \geq d(u, v) \geq 3$. Say $S$ has every vertex incident with $[S, \overline{S}]$. Thus $|S| \leq \left|[S, \overline{S}]\right| = \lambda(G) \leq \delta(G)$. Each vertex in $S$ is incident with at most $|S| - 1$ edges in $G[S]$, and so at least $\delta(G) - |S| + 1$ edges in $[S, \overline{S}]$. Thus

$$\lambda(G) = \left|[S, \overline{S}]\right| \geq |S| (\delta(G) - |S| + 1).$$

This last expression attains its minimum value of $\delta(G)$ when $|S| = 1$ or $|S| = \delta(G)$. In both cases we have $\lambda(G) \geq \delta(G)$, so $\lambda(G) = \delta(G)$. ■

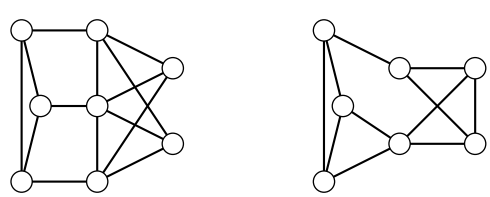
The following corollary follows from the proof of this theorem.

**Corollary 2.** [1] *If $G$ has diameter 2, then one of the subgraphs on one side of a minimum edge cut is either $K_1$ or $K_{\delta(G)}$.*

A **trivial edge cut** is an edge cut whose deletion isolates a single vertex. To study those diameter 2 graphs with a nontrivial minimum edge cut, we define the following set of graphs.

**Definition.** Let $\mathbb{G}$ be the set of graphs that contains the Cartesian product $K_{\frac{n}{2}} \square K_2$, $n \geq 4$, and those graphs with the following structure. The vertices can be partitioned into three sets, $S_1$, $S_2$, and $S_3$. Set $S_1$ induces $K_d$, $S_2$ has $n_2 \leq d$ vertices, and $S_3$ has $n_3$ vertices, $n_2 + n_3 > d$. There are $d$ edges joining a vertex of $S_1$ and a vertex of $S_2$ so that each vertex in $S_1 \cup S_2$ is incident with at least one edge. All possible edges between $S_2$ and $S_3$ are present. There are enough extra edges with both ends in $S_2$ or $S_3$ so that $\delta(G) \geq d$.

In the examples of graphs in $\mathbb{G}$ below, the sets $(S_1, S_2, S_3)$ induce graphs $(K_3, P_3, \overline{K}_2)$ at left and $(K_3, \overline{K}_2, K_2)$ at right.



**Theorem 3.** *A graph has diameter 2 and contains a non-trivial minimum edge cut if and only if it is in set* $\mathbb{G}$.

***Proof.*** ($\Leftarrow$) It is readily checked that a graph $G \in \mathbb{G}$ has $\delta(G) = d = \lambda(G)$, and contains a nontrivial minimum edge cut. Each graph $G$ has diameter 2 since each pair of vertices in $S_1$ and $S_3$ has a unique common neighbor.

($\Rightarrow$) Let $G$ have diameter 2 and contain a non-trivial minimum edge cut $X = \left[S_1, \overline{S}_1\right]$, and let $d = \delta(G)$. Then (say) $G[S_1] = K_d$, and the order of $\overline{S}$ is at least $d$. If it is exactly $d$, then $G = K_{\frac{n}{2}} \square K_2$.

If not, then $\overline{S}$ contains vertices not adjacent to any vertex of $K_d$. Let $S_3$ be the set of these vertices and $S_2 = \overline{S}_1 - S_3$. Then each vertex of $S_2$ is incident with at least one edge of $X$, and each vertex of $S_1$ is incident with exactly one edge of $X$. Then each vertex of $S_3$ is adjacent to each vertex of $S_2$ since otherwise some pair of vertices in $S_1$ and $S_3$ will have distance more than 2. Since $\delta(G) = d$, there are enough extra edges with both ends in $S_2$ or $S_3$ so that each vertex has degree at least $d$. ■

**Corollary 4.** *If* $G \in \mathbb{G}$, *it has between* $d$ *and* $\max\{n-1, 3d-1\}$ *trivial minimum edge cuts.*

***Proof.*** The number of trivial minimum edge cuts is the number of vertices of minimum degree. All the vertices of $K_d$ have minimum degree, so this is at least $d$. Now $K_{\frac{n}{2}} \square K_2$ has $n = 2d$ such vertices. If $G$ is regular, then it has at most $d + d + (d-1)$ vertices since each vertex in $S_2$ has degree at least $1 + |S_3|$. If $|S_3| \geq d$ then each vertex in $S_2$ has degree more than $d$, so there are at most $n - 1$ minimum degree vertices. ■

**Theorem 5.** *All graphs in set* $\mathbb{G}$ *have a single non-trivial minimum edge cut except for* $C_4$ *and those constructed as follows. Let a vertex* $v$ *be adjacent to* $s$, $\frac{d}{2} \leq s \leq d$, *vertices each in two copies of* $K_d$, $d \geq 2$, *and add a matching between* $d - s$ *vertices in each* $K_d$ *not adjacent to* $v$.

For $d = 2$, the three possible graphs in $\mathbb{G}$ with more than one non-trivial minimum edge cut are $C_4$, $C_5$, and $K_1 + 2K_2$.

***Proof.*** Let $G \in \mathbb{G}$, so $\delta(G) \geq 2$. Let $\delta(G) = 2$ and $|S_2| = 2$. Note that $C_4$ and $C_5$ have two and five nontrivial edge cuts, respectively. Now $C_5 + e$ has a single non-trivial minimum edge cut. Let $u$ and $v$ be the vertices in $S_2$. If there are at least two vertices in $S_3$, then $G$ has a spanning subgraph with $n-4$ $u-v$ paths of length 2 and one $u-v$ path of length 3. Hence the result holds for $\delta(G)=2$.

Let $\delta(G)=2$, $|S_2|=1$, and $v\in S_2$. If there is another nontrivial edge cut, it must separate $S_1\cup v$ from $K_2$ (by Corollary 2). Thus $G=K_1+2K_2$.

Let $d=\delta(G)>2$. Then no nontrivial minimum edge cut separates vertices in $K_d$. Assume there is another nontrivial edge cut $X$. One component of $G-X$ contains all vertices of $S_1$ and at least one of $S_2$. Thus the other component of $G-X$ must be $H=K_d$ by Corollary 2. Now there are $s\le d$ vertices of $H$ in $S_3$ and $d-s$ vertices of $H$ in $S_2$. If there are $r$ other vertices in $S_2$, then $X$ contains at least $rs+s-d\ge d$ edges. Equality requires $r=1$, so let $v$ be the one vertex in $S_2-H$. Also, each vertex in $S_2-v$ is adjacent to exactly one vertex of $S_1$. Then $v$ is adjacent to exactly $s$ vertices in $S_1$, so $s\ge\frac{d}{2}$. Then $G$ can be contructed as described and has exactly two non-trivial minimum edge cuts. ∎

Finally, we consider the nature of minimum edge cuts in almost all graphs.

**Theorem 6.** *Almost all graphs have a single minimum edge cut, which is trivial.*

***Proof.*** In random graph theory, it is known that almost all graphs have diameter 2 [1]. This implies that $\lambda(G)=\delta(G)$ for almost all graphs. Erdos and Wilson [2] showed that almost all graphs have a unique vertex of maximum degree. By symmetry, almost all graphs have a unique vertex of minimum degree.

Those graphs with a minimum non-trivial edge cut have the structure described in Theorem 3, including at least $\delta(G)>1$ vertices of minimum degree. Hence almost all graphs have a single minimum edge cut, which is trivial. ∎

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