EULER'S FORMULA

The Taylor series centered at 0 for e^x, $\sin x$, $\cos x$ are as follows.

$$\begin{array}{lcccccccccccccccccc}
e^x & = & 1 & + & x & + & \frac{x^2}{2} & + & \frac{x^3}{6} & + & \frac{x^4}{24} & + & \frac{x^5}{120} & + & \frac{x^6}{720} & + & \frac{x^7}{5040} & + & ... \\
\sin x & = & & & x & & & - & \frac{x^3}{6} & & & + & \frac{x^5}{120} & & & - & \frac{x^7}{5040} & + & ... \\
\cos x & = & 1 & & & - & \frac{x^2}{2} & & & + & \frac{x^4}{24} & & & - & \frac{x^6}{720} & & & + & ...
\end{array}$$

All the terms of the Taylor series for e^x appear as terms for sine and cosine, except that some signs are different. It appears that there may be some relationship between the functions. To account for the signs, we substitute ix into e^x, where $i = \sqrt{-1}$ is an imaginary number. Note that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, ... We find

$$e^{ix} = 1 + ix - \frac{x^2}{2} - i\frac{x^3}{6} + \frac{x^4}{24} + i\frac{x^5}{120} - \frac{x^6}{720} - i\frac{x^7}{5040} + ...$$

Separating the real and imaginary parts, we see

$$e^{ix} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + ...\right) + i\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + ...\right)$$

Thus we find Euler's Formula

$$e^{ix} = \cos x + i \sin x$$

We can interpret x as an angle in the complex plane, so that e^{ix} yields points on the unit circle.

EULER'S IDENTITY Plugging in $x = \pi$, we find $e^{i\pi} = -1$, or

$$e^{i\pi} + 1 = 0$$

This is called Euler's Identity. It involves five famous numbers:

0, the additive identity

1, the multiplicative identity

π, which comes from the geometry of circles

i, which comes from solving quadratic equations

e, which comes from calculus

The latter three, in particular, come from different branches of mathematics, and it is not obvious that they should have anything to do with each other. Yet they are all related in this one fundamental identity. This is an example of beauty in mathematics. One survey of mathematicians rated Euler's Identity the most beautiful equation in mathematics.

TRIG IDENTITIES Euler's Formula can give us insight into trigonometric identities. Consider substituting an angle sum $x+y$. We find

$$\begin{aligned}
\cos(x+y) + i\sin(x+y) &= e^{i(x+y)} \\
&= e^{ix}e^{iy} \\
&= (\cos x + i\sin x)(\cos y + i\sin y) \\
&= \cos x\cos y + \cos x \cdot i\sin y + i\sin x\cos y - \sin x\sin y \\
&= (\cos x\cos y - \sin x\sin y) + i(\cos x\sin y + \sin x\cos y)
\end{aligned}$$

Separating the real and imaginary parts, we see

$$\begin{aligned}
\cos(x+y) &= \cos x\cos y - \sin x\sin y \\
\sin(x+y) &= \cos x\sin y + \sin x\cos y
\end{aligned}$$

These are the angle sum identities for sine and cosine! They can be proved using geometric arguments, but they are difficult to remember. Yet they follow immediately from Euler's formula.

DIFFERENTIAL EQUATIONS Suppose we want to solve a differential equation of the form $y'' + ay' + by = 0$ for all possible solutions $y(t)$. We may suspect that there is a solution of the form $y = e^{rt}$. Substituting, we find $r^2e^{rt} + are^{rt} + be^{rt} = 0$, so $r^2 + ar + b = 0$. This is the characteristic equation of the differential equation. We can factor it to solve for the roots. If there are two real roots r_1 and r_2, the general solution is $y(t) = Ae^{r_1t} + Be^{r_2t}$.

But what if we have a differential equation whose characteristic equation has two complex roots $r = a \pm bi$? Such a differential equation has the form $y'' - 2ay' + (a^2+b^2)y = 0$. How can we obtain real solutions from the complex solution e^{a+bi}? We use Euler's Formula to see

$$e^{(a+bi)t} = e^{at}e^{bit} = e^{at}(\cos bt + i\sin bt) = e^{at}\cos bt + ie^{at}\sin bt$$

It is not hard to check that the real and imaginary parts are both solutions to $y'' - 2ay' + (a^2+b^2)y = 0$.

Example. The differential equation $y'' - 2y' + 5 = 0$ has characteristic equation $r^2 - 2r + 5 = 0$, with roots $r = 1 \pm 2i$. We have $e^{(1+2i)t} = e^t(\cos 2t + i\sin 2t) = e^t\cos 2t + ie^t\sin 2t$, so the general solution is

$$y(t) = Ae^t\cos 2t + Be^t\sin 2t$$