FINDING ZEROS WITH CALCULUS

For many functions, it is possible to find the zeros of a function by factoring or other algebraic means. However, for many other functions, this is difficult or impossible. Calculus helps to address this problem.

1 The Intermediate Value Theorem

Theorem. *The Intermediate Value Theorem (IVT). If a function $f(x)$ is continuous on $[a,b]$ and L is between $f(a)$ and $f(b)$, then there is at least one number $c \in (a,b)$ with $f(c) = L$.*

Proof. [sketch] For simplicity, we assume $L = 0$. Let $f(a) > 0 > f(b)$. There is a largest $c \in (a,b)$ such that $f(c) \geq 0$. Thus $f(x) \leq 0$ on $(c,b]$. If $f(c) > 0$, then $f(x) > 0$ on $(c-\delta, c+\delta)$, $\delta > 0$ since f is continuous. This is a contradiction. Thus $f(c) = 0$.

□

If we let $L = 0$, the IVT says that if a continuous function has opposite signs at a and b, then it has at least one zero in between.

Example. Let $f(x) = x^5 - 10x + 2$. This is a continuous function, and $f(0) = 2$ and $f(1) = -7$. Thus the IVT says it has a zero in $(0,1)$.

Any polynomial $p(x)$ is continuous at every point of its domain. If $p(x)$ is an odd polynomial and its leading coefficient is positive, then $\lim_{x\to\infty} p(x) = \infty$ and $\lim_{x\to-\infty} p(x) = -\infty$. If the leading coefficient is negative, the signs reverse. Either way, sign changes at some point. By the Intermediate Value Theorem, $p(x)$ must have a zero. This is an important piece of the proof of the Fundamental Theorem of Algebra, which says that every non-constant polynomial with complex coefficients has a complex zero.

The IVT tells us that a zero exists, but not where exactly it is. How can we find, or at least approximate, the zero? We can check the midpoint of the interval $\frac{a+b}{2}$. If it is the zero, we are done. If not, $f\left(\frac{a+b}{2}\right)$ it is either positive or negative. Then the interval $[a,b]$ is divided into two subintervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$. The function changes sign on exactly one of them. This narrows down the interval containing the zero. Repeating this process, we have the Bisection Method, summarized below.

Algorithm. *The Bisection Method.*

Given a continuous function $f(x)$ and interval $[a,b]$ where $f(a)$ and $f(b)$ have opposite signs,

1. If $\frac{a+b}{2}$ is the zero, stop.

2. If not, determine on which of the two subintervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ the function changes signs.

3. Repeat the process using this subinterval.

Example. Let $f(x) = x^5 - 10x + 2$. The results of the Bisection Method are summarized in the table below. Thus the zero is in $\left[\frac{3}{16}, \frac{7}{32}\right]$.

x	0	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{7}{32}$
$f(x)$	+	-	-	-	+	+	-

This is called the Bisection Method since the length of the interval is halved with each iteration of the algorithm. The advantage of the method is that it will always converge to a zero when applied correctly. The disadvantage is that it is relatively slow to do so. We will see that Newton's Method below is much faster.

2 Derivatives and Zeros

The IVT can guarantee at least one zero in an interval, but there is no limit to how many zeros there could be in an interval. For example, $\sin\left(\frac{1}{x}\right)$ has infinitely many zeros in $(0,1)$. We need a way to bound how many zeros a function has. We employ Rolle's Theorem.

Theorem. *Rolle's Theorem. Let $f(x)$ be differentiable on $[a,b]$. If $f(a)=f(b)$, then there is at least one point $c\in(a,b)$ with $f'(c)=0$.*

The contrapositive of a logical implication $P\Rightarrow Q$ is *not* $Q \Rightarrow$ *not* P. An implication and its contrapositive are logically equivalent, that is, they are both true or both false. We will employ the contrapositive of Rolle's Theorem.

Corollary. *Let $f(x)$ be differentiable on $[a,b]$. If $f'(c)\neq 0$ for any $c\in(a,b)$, then $f(a)\neq f(b)$.*

Thus if the derivative of a function is never zero on an interval, the function has at most one zero on the interval. Employing another theorem, this can be stated another way.

Corollary. *Let $f(x)$ be differentiable on $[a,b]$. If f is increasing (or decreasing) on $[a,b]$, then f has at most one zero on $[a,b]$.*

This should be intuitively reasonable when considering the graph of a function.

Example. Let $f(x)=x^3+5x+2$. Then $f'(x)=3x^2+5>0$. Thus f is increasing everywhere, so it has at most one zero. The IVT shows that there is exactly one zero.

Exercises.

For the equations in 1-3, show that they

a. have at least one solution

b. have at most one solution

1. $x^3+4x-2=0$
2. $\tan x-1=0$, $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
3. $\cos x=x$
4. Use the Bisection Method on the function $f(x)=x^2-2$ to find an interval with length at most $\frac{1}{8}$ with rational endpoints containing $x=\sqrt{2}$.

3 Newton's Method

We have seen that the Bisection Method converges slowly. Also, it can't be used when there is no change of signs around a zero, as in $f(x) = (x-2)^2$. When we can't find a zero exactly, we need a method to approximate it that converges quickly.

We begin Newton's Method with an initial guess x_0, hopefully near the zero. Then we find the tangent line to the curve at the point $(x_0, f(x_0))$. Then we find where the tangent line crosses the x-axis. We let this x-value be the new estimate x_1 and repeat the process.

The tangent line is

$$y = f(x_n) + f'(x_n)(x - x_n)$$

We set this equal to 0 and solve, obtaining

$$f'(x_n)(x - x_n) = -f(x_n)$$

$$x_{n+1} = x = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is the formula for Newton's Method.

Example. Newton's Method can be used to find approximations for zeros. If we want to approximate $x = \sqrt{2}$, we need a polynomial with rational coefficients with this zero. Since $x^2 = 2$, $f(x) = x^2 - 2$ has $\sqrt{2}$ as a zero. Newton's Method yields

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$$

We begin with a guess of $x_0 = 2$. Then

$$x_1 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$x_2 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12}$$

$$x_3 = \frac{17}{24} + \frac{12}{17} = \frac{577}{408}$$

We can continue this process indefinitely, but the arithmetic becomes increasingly tedious. Using a calculator would be productive here. We can make the iteration of Newton's Method simple as follows. First, enter the initial guess. Then enter the formula for Newton's Method applied to the specific function f, with the previous answer command ANS in place of the variable. For the previous example, enter ANS/2+1/ANS. Then hitting enter produces the next estimate, and hitting enter repeatedly produces a sequence of estimates. This produces

n	estimate x_n
0	2
1	1.5
2	1.416666667
3	1.414215686
4	1.414213562
5	1.414213562

The last two estimates should not be exactly equal, but they are so close that the calculator rounds them identically. Newton's Method has converged to a zero. Indeed, $\sqrt{2} = 1.414213562....$

We see that Newton's Method converged quickly in this example. It took only five iterations to achieve nine digits of accuracy. Indeed, this is usually the case.

Let $f(x)$ be a function with f'' continuous, $f(a) = 0$, and $f'(a) \neq 0$. If Newton's Method starts "close enough" to a, then the number of accurate digits of the approximation will roughly double.

Newton's Method usually works well and converges quickly to a zero. However, there are a number of things that can go wrong in unusual cases. The following functions and initial values provide a number of interesting examples to investigate.

Function	Initial Value(s)	What Goes Wrong?
$y = (x-1)^2 + .01$	$x_0 = .5$	no zero
$y = x^2 - x$	$x_0 = .5$	horizontal tangent
$y = \sqrt[3]{x}$	$x_0 = 1$	diverges
$y = \sqrt{\lvert x\rvert}$	$x_0 = r > 0$	oscillates
$y = \sin x$	$x_0 = 2, 1.8, 1.6$	converges to wrong zero
$y = \arctan x$	$x_0 = 1.39, 1.4$	too far away
$y = \frac{4x^2}{1+4x^2} - .16$	$x_0 = 1, 1+\epsilon, 1-\epsilon$	chaotic near $x = 1$
$y = (x-1)^{40}$	$x_0 = 2$	converges slowly, gets stuck
$y = \pi + 2x \sin \frac{\pi}{x}$	$x_0 = .5$	converges, but no zero

Exercises.

Apply Newton's Method to the following functions with the given starting values. Do one iteration by hand, then use a calculator.

5. $f(x) = x^2 - 3$, $x_0 = 3$
6. $f(x) = x^5 - 10x + 2$, $x_0 = 1$
7. $f(x) = \tan x$, $x_0 = 3$
8. $f(x) = 2x - \tan x$, $x_0 = 1$