GROWTH RATES OF FUNCTIONS

Many functions go to infinity when their input goes to infinity. This includes x^2, $\ln x$, and 2^x. We often need to know how fast a function grows. One way to do this is to calculate many terms of the functions and compare them.

x	x^2	$\ln x$	2^x
1	1	0	2
10	100	2.303	1024
100	10000	4.605	$\approx 10^{30}$
1000	1000000	6.908	$\approx 10^{301}$
10000	10^8	9.210	$\approx 10^{3010}$
100000	10^{10}	11.513	$\approx 10^{30103}$
1000000	10^{12}	13.816	$\approx 10^{301030}$
10000000	10^{14}	16.118	$\approx 10^{3010300}$
100000000	10^{16}	18.421	$\approx 10^{30103000}$

It appears that 2^x grows faster than x^2, which grows faster than $\ln x$. However, this is not a mathematically precise argument. Also, we can never be sure how many terms must be calculated before a pattern become apparent. Instead, we can compare two functions by examining the ratio of their terms as x grows large.

Definition 1. A function $f(x)$ **grows faster than** $g(x)$ (or g grows slower than f) if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \text{ } or \text{ } \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

In this case we write $f \gg g$ or $g \ll f$.

Functions $f(x)$ and $g(x)$ **grow at the same rate** if for some L, $0 < L < \infty$,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

Example. Compare the following pairs of functions and determine which grows faster.
a. $\ln x$ and x
b. $\ln x$ and $\ln \ln x$
c. e^x and x^p, $p > 1$
Solution. We find the limit of the ratio of functions, using L'Hopital's Rule as necessary.

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

$$\lim_{x\to\infty} \frac{\ln \ln x}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{x \ln x}}{1/x} = \lim_{x\to\infty} \frac{1}{\ln x} = 0$$

$$\lim_{x\to\infty} \frac{e^x}{x^p} = \lim_{x\to\infty} \frac{e^x}{px^{p-1}} = \lim_{x\to\infty} \frac{e^x}{p(p-1)x^{p-2}} = \ldots = \infty$$

Thus we see $\ln \ln x \ll \ln x \ll x$, and $x^p \ll e^x$. Thus any power function grows slower than e^x (or any exponential function with base more than 1). Many common functions can be ordered by their growth rates, as follows.

Theorem 2.

$$\ln \ln x \ll \ln x \ll \underset{0<p<1}{x^p} \ll x \ll x \ln x \ll \underset{p>1}{x^p} \ll e^x \ll x^x \ll e^{x^2}$$

Additional functions can be inserted into this order. For example, additional exponential functions $(1.1)^x \ll 2^x \ll e^x \ll 3^x$ could be inserted in place of e^x. The polynomial functions $x^2 \ll x^3 \ll x^4 \ll x^5$ could be inserted. Not every function fits into this order; one could construct a function that crosses more than one of these function infinitely many times. However, these are the most common function used in calculus.

Functions can be very complicated. However, often an approximation is all we need. The following notation is called **big-O notation**.

Definition 3. Let f and g be functions. We say $g \in \mathcal{O}(f)$ if $|g(x)| \leq c\,|f(x)|$ for some c and sufficiently large x.

Exercises.

1. Use limits to verify the growth rate comparisons from Theorem 2.

a. $\ln x \ll \underset{0<p<1}{x^p}$

b. $x \ln x \ll \underset{p>1}{x^p}$

c. $e^x \ll x^x$

2. Use limits to determine which function grows faster, or if they grow at the same rate.

a. x^{1000}, e^x

b. $\ln \ln x^2$, $\ln \ln \sqrt{x}$

c. e^{n^2}, $e^{n \ln n}$

d. $\sqrt{x^2-6}$, $\sqrt[3]{1+x^6}$

3. Use limits to put the following functions in order of their growth rate. Where do they fit in the order of functions in Theorem 2?

a. $x^3 2^x$, $x^2 3^x$, $x^4 e^x$

b. $x(\ln x)^2$, $x^2 \ln x$, $x \ln(x^2)$

c. $\ln(x \ln x)$, $\ln \ln 2^x$, $\ln \ln \ln x$

d. $e^{\sqrt{x}}$, x^x, e^{x^2}

4. Use limits to determine which functions are in $\mathcal{O}(f)$ for the given f.

a. $x+3$, x^2+4x, x^2-x^3, $\sqrt{x^4+5}$; $f=x^2$

b. $\log_2 x$, $\log x$, $\sqrt{x}$, $\ln x^2$; $f=\ln x$

c. 3^x, $4^{x/2}$, x^x, $e^{x+\frac{1}{x}}$; $f=e^x$