EVALUATING INFINITE LIMITS

Some limits do not exist (as real numbers) because the function goes to infinity. When a function $f(x)$ goes to infinity as $x \to a$, it is usually the case that trying to plug in a results in division by 0. To determine what f does near a, we can break down the function into successive operations and determine what the limit does to each operation.

Example. Determine what happens to $f(x) = -\frac{1}{x-3}$ as $x \to 3^-$, $x \to 3^+$, and $x \to 3$.

Strategy. When a number is substituted for x, we successively perform the following operations: subtract 3, divide into 1, negate. Starting with what happens to x, we add one more operation in each step until we know what happens to $f(x)$. Remember that $x \to a^-$ means that $x - a$ is a small negative number, and $x \to a^+$ means that $x - a$ is a small positive number.

Solution. Assume $x \to 3^-$. Then $x - 3 \to 0^-$. Dividing a small number into a number produces a number that is large (in absolute value), so $\frac{1}{x-3} \to -\infty$. Thus $-\frac{1}{x-3} \to \infty$. Assuming that $x \to 3^+$, a similar argument shows that $x - 3 \to 0^+$, $\frac{1}{x-3} \to \infty$, and $-\frac{1}{x-3} \to -\infty$. We conclude that $\lim_{x\to 3^-} f(x) = \infty$ and $\lim_{x\to 3^+} f(x) = -\infty$. Since $f(x)$ goes different places from two sides, the limit as $x \to 3$ does not exist.

Example. Determine what happens to $f(x) = \frac{x^2-4}{(x-2)^3}$ as $x \to 2^-$, $x \to 2^+$, and $x \to 2$.

Solution. Note that we can simplify the function:

$$f(x) = \frac{x^2-4}{(x-2)^3} = \frac{(x+2)(x-2)}{(x-2)^3} = \frac{(x+2)}{(x-2)^2}.$$

While x occurs more than once in the function, only the occurrence in the denominator causes a discontinuity. When $x \approx 2$, $x + 2 \approx 4$. We summarize our reasoning in the following table.

$x \to 2^-$	$x \to 2^+$
$x - 2 \to 0^-$	$x - 2 \to 0^+$
$(x-2)^2 \to 0^+$	$(x-2)^2 \to 0^+$
$\frac{(x+2)}{(x-2)^2} \to \infty$	$\frac{(x+2)}{(x-2)^2} \to \infty$

Note that the square (or any even power) of a negative number is positive, while any odd power of a negative number is still negative.

Recall that an asymptote is a line that a function approaches arbitrarily closely. More specifically, $f(x)$ has a vertical asymptote $x = a$ if $\lim_{x\to a^\pm} f(x) = \pm\infty$. The reasoning used above can help to sketch the graph of a function near its asymptotes.

Example. Determine what happens to $f(x) = \frac{x+3}{(x+2)^2(x-5)}$ near its asymptotes.

Solution. Since division by zero causes asymptotes, we suspect that f has asymptotes at $x = -2$ and $x = 5$. Starting with $x = 5$, note that $\frac{x+3}{(x+2)^2}$ is positive near 5. The table below left summarizes our reasoning.

$x \to 5^-$	$x \to 5^+$
$x - 5 \to 0^-$	$x - 5 \to 0^+$
$f(x) \to -\infty$	$f(x) \to \infty$

$x \to -2^-$	$x \to -2^+$
$x + 2 \to 0^-$	$x + 2 \to 0^+$
$(x+2)^2 \to 0^+$	$(x+2)^2 \to 0^+$
$f(x) \to -\infty$	$f(x) \to -\infty$

Near -2, $\frac{x+3}{x-5}$ is negative. The table above right summarizes our reasoning. Thus $x = -2$ and $x = 5$ are both vertical asymptotes.

Exercises.

Evaluate the following limits for $f(x)$ as $x \to a^-$, $x \to a^+$, and $x \to a$.

1. $(a = 1)$ $f(x) = \frac{3}{x-1}$
2. $(a = 4)$ $f(x) = -\frac{5}{(x-4)^2}$
3. $(a = -2)$ $f(x) = \frac{x}{(x+2)^3}$
4. $(a = -3)$ $f(x) = \frac{x-7}{(x+3)^4}$

Determine what $f(x)$ does near each of its asymptotes.

5. $f(x) = \frac{x}{x^2-4}$
6. $f(x) = \frac{2-x}{x^3-4x}$
7. $f(x) = \frac{x^2-16}{x^2-8x+15}$
8. $f(x) = \frac{5-x}{x^3-10x^2+21x}$