THE INTEGRAL TEST

Most infinite series cannot be easily evaluated. We need tests to determine whether they converge or diverge. If the series diverges, there is no need to analyze it further. If it converges, we can try to find an exact value, or estimate it.

An infinite series may resemble a Riemann sum, so we can sometimes compare it to an integral to determine convergence or divergence.

Theorem 1. (**The Integral Test**) *If $f(x)$ is a continuous, positive, decreasing function, and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.*

Proof. Since $f(x)$ is decreasing, we have the following inequalities.

$$\sum_{k=2}^{n} a_k \le \int_1^n f(x)\,dx \le \sum_{k=1}^{n-1} a_k$$

The lower bound is a Riemann sum using the right endpoints. The upper bound is a Riemann sum using the left endpoints. The inequalities hold for all n, as $n \to \infty$. Thus

$$\sum_{k=2}^{\infty} a_k \le \int_1^{\infty} f(x)\,dx \le \sum_{k=1}^{\infty} a_k$$

so the integral and series either both converge or both diverge.

□

Example. Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges.

Solution. Consider the sum as a Riemann sum. Chopping off the first term, we see $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a lower Riemann sum for $\int_1^{\infty} \frac{1}{x^2}dx$. Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2}dx = 1 + 1 = 2.$$

Thus the sum converges to a value that is less than 2.

Example. Determine when the p-**series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Solution. Using the Integral Test, we consider the integral $\int_1^{\infty} \frac{1}{x^p}dx$. Note the antiderivative $\int \frac{1}{x^p}dx = \frac{1}{-p+1}x^{-p+1}$ when $p \neq 1$.

When $p > 1$, x^{-p+1} goes to 0 as $x \to \infty$. In this case, we have convergence: $\int_1^{\infty} \frac{1}{x^p}dx = 0 - \frac{1}{-p+1} = \frac{1}{p-1}$.

When $p < 1$, $x^{-p+1} \to \infty$ as $x \to \infty$. Thus we have divergence: $\int_1^{\infty} \frac{1}{x^p}dx = \infty$.

When $p = 1$, we find $\int_1^{\infty} \frac{1}{x}dx = \ln|x||_1^{\infty} = \infty$. Thus by the Integral Test, the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \le 1$. The case $p = 1$ is the Harmonic Series again. It is the boundary case between convergence and divergence for the p-series.

Example. Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n}$ converges or diverges.

Solution. Note that $\frac{1}{n \cdot \ln n} < \frac{1}{n}$, so an easy comparison does not answer the question. Using the Integral Test, we consider the integral $\int_2^{\infty} \frac{1}{x \cdot \ln x}dx$. Using substitution, we find

$$\int_2^{\infty} \frac{1}{x \cdot \ln x}dx = \ln\ln x|_2^{\infty} = \infty.$$

Thus the series diverges.

Since the function $\ln\ln x$ grows very slowly, the series $\sum_{n=2}^{\infty}\frac{1}{n\cdot\ln n}$ diverges very slowly. How slowly? Let's estimate how many terms must be summed to get a sum of 10. This will occur approximately when $\ln\ln n = 10$. Solving, we find $n = e^{e^{10}}$. How big is that?

To estimate the size of a large number N, write it in scientific notation: $N = a\cdot 10^b$. To find the number of digits b, we could take a logarithm. Now $\log N = \log a + b$, so there are $\lfloor \log N \rfloor$ digits in the decimal representation of N.

Now we see $\log n = \log e^{e^{10}} = e^{10}\log e \approx 9566$. Thus $n \approx 10^{9566}$, so n has about 9566 digits. Some students may be tempted to determine divergence by adding a bunch of terms and seeing whether the result is a large number. But this method fails when the sum diverges slowly, as in this case. Adding the first billion (10^9) terms won't even get close to double digits! Convergence tests are necessary.

(*) Example. Consider the convergence of

$$\sum_{p\,prime}\frac{1}{p} = \frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+...$$

where the sum ranges over all prime numbers p. Recall that a prime number is a natural number greater than 1 that is not a product of two smaller natural numbers. To determine convergence or divergence, we need to know how prime numbers are distributed. The **Prime Number Theorem** provides the answer. Let p_n be the n^{th} prime number. It says $\lim\limits_{n\to\infty}\frac{p_n}{n\cdot\ln n} = 1$. This implies that $p_n \approx n\cdot\ln n$. Thus $\sum\limits_{p\,prime}\frac{1}{p} \approx \sum\frac{1}{n\cdot\ln n} = \infty$.

Example. Determine when the **logarithmic p-series** $\sum_{n=1}^{\infty}\frac{1}{n(\ln n)^p}$ converges.

Solution. Note that the case $p = 1$ was handled in a previous example. Note that when $p \neq 1$, $\int \frac{1}{x(\ln x)^p}dx = \int \frac{1}{u^p}du$, using the substitution $u = \ln x$. Since $\ln x \to \infty$ as $x \to \infty$, the result is the same as for the p-series. The logarithmic p-series converges when $p > 1$ and diverges when $p \le 1$.

By generalizing this example, we can find a sequence of functions whose improper integrals diverge more and more slowly, and a sequence whose improper integrals converge to larger and larger values. The corresponding series approach a line between convergence and divergence.

$$\sum_{n=K}^{\infty}\frac{1}{n^{1+\epsilon}} < \sum_{n=K}^{\infty}\frac{1}{n\left(\ln n\right)^{1+\epsilon}} < \sum_{n=K}^{\infty}\frac{1}{n\left(\ln n\right)\left(\ln\ln n\right)^{1+\epsilon}} < \underset{conv}{\cdots} \parallel \underset{div}{\cdots} < \sum_{n=K}^{\infty}\frac{1}{n\left(\ln n\right)\ln\ln n} < \sum_{n=K}^{\infty}\frac{1}{n\left(\ln n\right)} < \sum_{n=K}^{\infty}\frac{1}{n}$$

(Note that $\epsilon > 0$ is any small positive number and K is any constant for which the logarithms are all defined.)

The Integral Test can also be used to estimate the value of a convergent series. Let $S = \sum_{n=1}^{\infty} a_n$ and $s_n = \sum_{i=1}^{n} a_i$. Define the remainder $R_n = S - s_n = a_{n+1} + a_{n+2} + ...$ Then

$$\begin{array}{rcccl} \int_{n+1}^{\infty} f(x)\,dx & \le & R_n & \le & \int_{n}^{\infty} f(x)\,dx \\ s_n + \int_{n+1}^{\infty} f(x)\,dx & \le & S & \le & s_n + \int_{n}^{\infty} f(x)\,dx \end{array}$$

Example. Estimate the value of $S = \sum_{n=1}^{\infty}\frac{1}{n^2}$ using the first five terms.

Solution. Note that $s_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \approx 1.4636$. Now $\int_n^{\infty}\frac{1}{x^2}dx = \frac{1}{n}$. Plugging in $n = 5$ and $n = 6$ produces the upper and lower bounds $s_5 + \frac{1}{6} \le S \le s_5 + \frac{1}{5}$. Thus $1.6303 \le S \le 1.6636$. Averaging the two bounds, we see $S \approx 1.645$. It is possible (but significantly more difficult) to show that $S = \frac{\pi^2}{6} \approx 1.644934$.