CHOOSING THE RIGHT INTEGRATION TECHNIQUE

There are many techniques for evaluating integrals. How can you pick the right one? There is no perfect rule for which technique to use, but the following guidelines should help.

ALGEBRA Algebraic techniques can be used to convert the integrand to a form in which it can be integrated. Such techniques include distributing a factored expression and breaking up a fraction. This helps since integrating a sum or difference is usually much easier than integrating a product or quotient. Examples are given below.

$$\int x\left(x^2+4\right)dx=\int\left(x^3+4x\right)dx$$

$$\int \frac{x^2+4}{x}dx=\int\left(x+\frac{4}{x}\right)dx$$

SUBSTITUTION Use substitution on an integral with the form $\int f\left(g\left(x\right)\right)g'\left(x\right)dx$. Substitute $u=g\left(x\right)$ and $du=g'\left(x\right)dx$. Keep in mind that $g'\left(x\right)$ must multiply $dx$for the substitution to work. If there is still an x in the integral, try resubstitution, that is, solve the substitution equation for x and substitute again.

INTEGRATION BY PARTS When the integrand is a product of two factors, integration by parts may be a good choice. At least one of the factors must be integrable, and the other should have a derivative that is simpler (or at least no more complicated) than it. Integration by parts sometimes needs to be applied more than once. If the same integral reappears, it may be possible to solve the equation for it.

TRIGONOMETRIC FUNCTIONS When the integrand involves trig functions, trig identities may be used to reexpress the integrand. These identities include reciprocals ($\sec x=\frac{1}{\cos x}$), ratios ($\tan x=\frac{\sin x}{\cos x}$), Pythagorean ($\sin^2 x+\cos^2 x=1$), double angle ($\sin 2x=2\sin x\cos x$) and power reducing ($\cos^2 x=\frac{1+\cos 2x}{2}$ and $\sin^2 x=\frac{1-\cos 2x}{2}$). For the integral $\int \sin^m x\cos^n x\,dx$, $m,n\in\mathbb{Z}$, there are ten different cases depending on whether m and n are positive or negative and odd or even. Most of these cases eventually require substitution; two use integration by parts.

TRIGONOMETRIC SUBSTITUTIONS Trig substitutions are used for integrals that contain $\sqrt{\pm a^2\pm x^2}$. A right triangle is used to replace all parts of the integral with trig functions involving an angle θ. Then an integral involving trig functions must be evaluated. Trig substitutions can also be used for integer powers of $\sqrt{\pm a^2\pm x^2}$.

POLYNOMIAL DIVISION To integrate a rational function, first check the degrees of the polynomials. If the degree of the numerator is larger or equal to the degree of the denominator, use polynomial division to break up the function into separate terms. If there is a remainder, it may need to be broken up using partial fractions.

PARTIAL FRACTIONS When a rational function has a smaller degree for its numerator than its denominator, it may need to be broken up using partial fractions. Equate it with the appropriate fractions, clear the denominators, and solve for the unknown constants. The partial fractions can be integrated using the power rule, ln, arctan, or substitution.

MULTIPLE TECHNIQUES Many integrals require the use of more than one integration technique. In particular, substitution and integration by parts always produce another integral. Solving it may require other techniques.

CONCLUSION Some integrals can be successfully integrated using more than one integration technique. Others have no simple antiderivative, and must be approached using numerical integration or power series. Students should practice integration with a mixed problem set to develop their ability to choose the right integration technique.