k-Paths of k-Trees

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Abstract

A k-tree is a graph that can be formed by starting with K_{k+1} and iterating the operation of making a new vertex adjacent to all the vertices of a k-clique of the existing graph. When the order $n > k+1$, a k-path graph is a k-tree with exactly two vertices of degree k. We state a forbidden subgraph characterization for k-paths as k-trees. We characterize k-trees with diameter $d \geq 2$ based on the k-paths they contain.

1 Introduction

In this paper, we seek to describe the structure of k-trees using k-paths, particularly focusing on the diameter of k-trees. Undefined notation and terminology will follow [2].

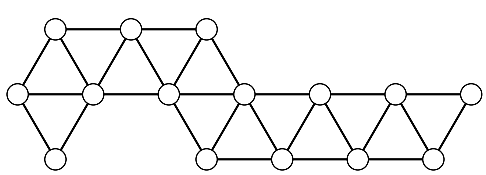
This work builds on previous papers on the Wiener index of maximal k-degenerate graphs [3] (with Zhongyuan Che) and on maximal k-degenerate graphs with diameter 2 [4].

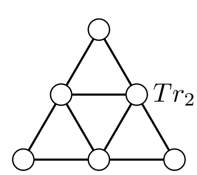
Definition 1. A k**-tree** is a graph that can be formed by starting with K_{k+1} and iterating the operation of making a new vertex adjacent to all the vertices of a k-clique of the existing graph. The clique used to start the construction is called the **root** of the k-tree.

A k**-leaf** is a degree k vertex of a k-tree.

A k**-path graph** G is an alternating sequence of distinct k- and $k+1$-cliques $e_0, t_1, e_1, t_2, ..., t_p, e_p$, starting and ending with a k-clique and such that t_i contains exactly two k-cliques e_{i-1} and e_i.

An example of a 2-path (which is also a 2-tree) is shown below left. A 2-tree that is not a 2-path (the triangular grid Tr_2) is below right.





Note that k-paths are also known as linear k-trees [1]. They are closely related to pathwidth [6]; in particular, they are the maximal graphs with proper pathwidth k. There is a simple characterization of these graphs.

Theorem 2. [5] *Let G be a k-tree with $n > k+1$ vertices. Then G is a k-path graph if and only if G has exactly two k-leaves.*

This leads to a forbidden subgraph characterization for k-paths as k-trees.

Theorem 3. *A k-tree is a k-path if and only if it does not contain $K_k + \overline{K}_3$ or for $k \geq 2$, $Tr_2 + K_{k-2}$.*

Proof. ($\Rightarrow$) (contrapositive) These graphs contain three k-leaves, so they are not k-paths.

($\Leftarrow$) (contrapositive) A k-tree that is not a k-path must have at least three k-leaves. Then it must contain a subgraph G that is minimal with respect to this property. It will have exactly three k-leaves, and deleting any of them results in a k-path. Let H be the graph formed by deleting all k-leaves from G. If H is not a clique, then it has two k-leaves, one of which has only one k-leaf of G neighboring it, so G is not minimal.

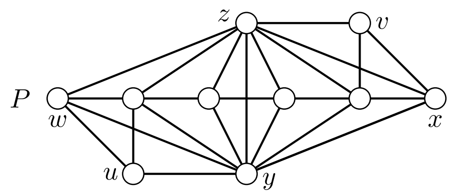
If $H = K_k$, $G = K_k + \overline{K}_3$. If $H = K_{k+1}$, each of its vertices are adjacent to a k-leaf of G. If two k-leaves of G have the same neighborhood, then G is not minimal. Thus there are $k-2$ vertices of H adjacent to all three k-leaves of G, and deleting them produces Tr_2. $\square$

2 Diameter of k-Trees

A tree is minimal with respect to diameter d if and only if it is P_{d+1}. In [4], I found a characterization of k-trees minimal with respect to diameter 3.

Definition 4. A **dominating vertex** of a graph is a vertex adjacent to all other vertices.

Algorithm 5. *Let P be a $k-2$-path, $k \geq 3$, of order $n-4$ with k-leaves w and x. Join dominating vertices y and z to P, forming $P + K_2$. Add u with neighborhood $N(w) \cup \{w, y\}$, and v with neighborhood $N(x) \cup \{x, z\}$. Let $\mathbb{G}_k$ be the class of all graphs formed this way.*



Theorem 6. [4] *A graph G is a k-tree minimal with respect to diameter 3 if and only if $G \in \mathbb{G}_k$.*

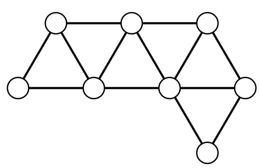
Equivalently, a k-tree has diameter at most 2 if and only if it does not contain any graph in $\mathbb{G}_k$.

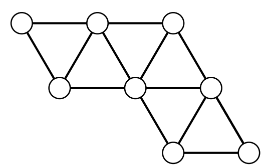
The graphs in $\mathbb{G}_k$ are all k-paths. A generalization also holds.

Lemma 7. *A k-tree minimal with respect to diameter $d \geq 2$ is a k-path.*

Proof. A k-tree with diameter at least d must contain a pair of vertices distance d apart. Now adding a vertex to a k-tree cannot change any existing distances. Thus in a minimal k-tree with diameter d, the vertices at distance d must be k-leaves, and no other vertices are k-leaves. □

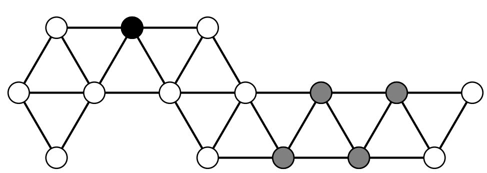
The 2-paths with diameter d cannot be characterized solely by their degree sequences, as there are two 2-paths with degree sequence 5,4,4,3,3,3,2,2 which have diameters 3 and 4 (see below). A characterization based on the arrangement of the degree 4 vertices is possible.





Definition 8. A **hub** is a vertex of degree at least 5 of a 2-path. A **truss** is a subgraph induced by vertices of degree 4 in a 2-path. An **external truss** has a vertex neighboring a 2-leaf, an **internal truss** does not.

In the 2-path below, the black vertex is an internal truss and the gray vertices induce an external truss.



Theorem 9. *Let G be a 2-tree minimal with respect to diameter d. Then G is a 2-path, and if $G \neq P^2_{2d}$, the 2-leaves are adjacent to external trusses with odd order. If h is the number of hubs, t_i is the order of the ith internal truss, and t' and t'' are the orders of the external trusses, then $d = h + \sum \left\lfloor \frac{t_i}{2} \right\rfloor + \left\lceil \frac{t'}{2} \right\rceil + \left\lceil \frac{t''}{2} \right\rceil + 1$.*

Proof. By Lemma 7, a minimal 2-tree with diameter d is a 2-path. To show the formula holds, we use induction on n. Since $G \neq P^2_{2d}$, it contains a hub. We start with the fan induced by its closed neighborhood. This has $h = 1$, $d = 2$, and all other quantities 0. We add vertices one at a time, checking that the formula holds in each case.

There are only two choices how to add a new 2-leaf next to an existing 2-leaf. In one choice, the other neighbor had degree at least 4. If it is already a hub, the diameter does not increase. If it is part of a truss of odd order, one vertex of the truss becomes a hub, the rest of the truss (if any) becomes internal, the sum does not change, and the diameter does not increase. If it is part of a truss of positive even order, one vertex of the truss becomes a hub, the rest of the truss becomes internal, the sum does not change, and the diameter does not increase.

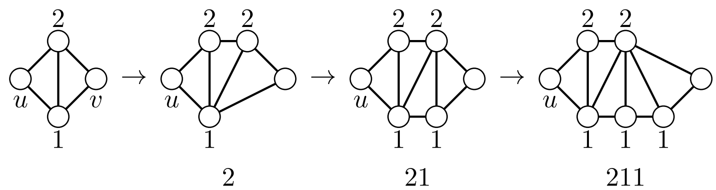
In the other choice, the other neighbor had degree 3, so we create an external truss or add one vertex to an existing external truss. If the truss had odd order, adding this vertex does not change the diameter. If the truss is new or had even order, adding this vertex increases the diameter by 1.

Since only the last case increases the diameter, in a 2-path minimal with respect to d, the 2-leaves are adjacent to external trusses with odd order. □

Thus a 2-tree with order $n \geq 5$ has diameter at least d if any only if it contains a 2-path with the properties described in the theorem. This implies that a 2-tree has diameter at least 3 if any only if it contains P^2_6.

To characterize k-trees with diameter d, we need a way to describe the construction of k-paths.

A k-path can be constructed from $K_k + \overline{K}_2$ with k-leaves u and v by maintaining u as a k-leaf and adding a new k-leaf adjacent to v and $k-1$ of its k neighbors. Label the k neighbors of u 1 through k (in any way). Each time a k-leaf x is added adjacent to (old) k-leaf w, label w with the label of its neighbor that does not neighbor x.



Define a string of length $n-k-2$ with the labels added after the first k. Call this a **construction string** of the k-path.

Definition 10. A string of numbers **contains a pattern** if the numbers in the pattern occur in order (not necessarily consecutively) in the string.

For example, the pattern 321 is contained in 312213 but not 132233.

Theorem 11. *A k-tree has diameter $d \geq 2$ if and only if it contains a k-path whose construction string contains at least $d-2$ consecutive permutations of $\{1, ..., k\}$.*

Proof. By Lemma 7, a k-tree with diameter d contains a k-path with diameter d. Let G be a k-path with diameter d and k-leaves (say) u and v. We show that the number of consecutive permutations of $\{1, ..., k\}$ in the string is always $d-2$. Certainly this is true for $K_k + \overline{K}_2$, which is minimal with diameter 2 and has an empty string.

Let H be a minimal k-path contained in G with k-leaves u and w. The vertices in $N(w)$ have labels $1, \ldots, k$. Each vertex added to form G removes one vertex from the neighborhood of the k-leaf it replaces, so at most one vertex from $N_H(w)$. To increase the diameter, each vertex in $N_H(w)$ must be removed, and each will be replaced with another vertex with the same label. The diameter increases by one exactly when the string contains one more permutation of $\{1, ..., k\}$. $\square$

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