Extremal Decompositions for Nordhaus-Gaddum Theorems

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Abstract

A Nordhaus-Gaddum theorem states bounds on $p(G)+p\left(\overline{G}\right)$ and $p(G) \cdot p\left(\overline{G}\right)$ for some graph parameter $p(G)$. We consider the sum upper bound for degeneracy, chromatic number, fractional and circular chromatic number, list chromatic number, span, and point partition number. Viewing $\left\{G, \overline{G}\right\}$ as a decomposition of K_n, we describe a strategy to determine the extremal decompositions for these parameters. This produces short proofs of several existing results as well as several new theorems.

Keywords: Nordhaus-Gaddum, decomposition, chromatic number, list coloring, degeneracy, L(2,1) labeling, vertex arboricity, point partition number

One common way to study a graph parameter $p(G)$ is to examine the sum $p(G)+p\left(\overline{G}\right)$ and product $p(G) \cdot p\left(\overline{G}\right)$. A theorem providing sharp upper and lower bounds for this sum and product is known as a **Nordhaus-Gaddum theorem**. The original Nordhaus-Gaddum Theorem [30] dealt with chromatic number; there are now hundreds of analogous results for other parameters. Of the four possible bounds, the sum upper bound has attracted the most attention.

Many authors state when a bound in a Nordhaus-Gaddum theorem is an equality in terms of conditions on a graph. This often leads to awkward characterizations, as authors try to describe conditions on $\overline{G}$ as conditions on G. It is more convenient to think of G and $\overline{G}$ as a decomposition of K_n.

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Definition 1. A **decomposition** of G is a set of nonempty subgraphs, called **factors**, whose edge sets partition $E(G)$. The subgraphs are said to **decompose** G. A k**-decomposition** of a graph G is a decomposition of G into k subgraphs. We use $\{G_1, G_2\}$ to denote a decomposition of G with factors G_1 and G_2.

Aouchiche and Hansen [1] compiled a huge survey of Nordhaus-Gaddum theorems for 2-decompositions (see also [8]). There are also Nordhaus-Gaddum theorems for decompositions with more than two factors. See [3] for degeneracy and [21] for degeneracy, chromatic number, and more.

We focus on the problem of determining extremal graphs for the upper bound when the parameter is degeneracy, chromatic number, fractional and circular chromatic number, list chromatic number, span, and point partition number. There is no strategy that works for any parameter, but we describe an approach that works for many related graph coloring parameters.

The basic strategy we use is to find a bound for the parameter (usually using degeneracy) and sum the bounds over the two factors. We also need a characterization of when the bound is an equality (such as Brooks' Theorem), at least under some restrictions. We consider the critical subgraphs and find the subdecomposition where they overlap, and use this to determine all extremal decompositions.

Definitions of terms and notation not defined here appear in [5]. In particular, K_n and C_n are the complete graph and cycle of order n, and $K_{1,n-1}$ is the star of order n. Also, $\overline{G}$ is the complement of G and $G + H$ is the join of graphs G and H. We use n for the number of vertices when the context is clear.

1. Degeneracy

Definition 2. A graph is k**-degenerate** if its vertices can be successively deleted so that immediately prior to deletion, each has degree at most k. The **degeneracy** $D(G)$ of a graph G is the smallest k such that it is k-degenerate. A **deletion sequence** of a graph G is a sequence of its vertices formed by iterating the operation of deleting a vertex of smallest degree and adding it to the sequence until no vertices remain. A graph G is k**-monocore** if $D(G) = \delta(G) = k$.

Equivalently, a graph is k-degenerate if $\delta(H) \leq k$ for every subgraph H (see also [4]).

Theorem 3. *For any graph G, $D(G) + D\left(\overline{G}\right) \leq n - 1$. The graphs for which $D(G) + D\left(\overline{G}\right) = n - 1$ are exactly the graphs constructed by starting with a regular graph and iterating the following operation.*
Given $k = D(G)$, H a k-monocore subgraph of G, add a vertex adjacent to at least $k + 1$ vertices of H, and all vertices of degree k in H (or similarly for $\overline{G}$).

A proof of the upper bound due to H. V. Kronk is implicit in [14]. It is stated explicitly without proof by Borodin [10]. The proof for the extremal decompositions appears in [3]. Borodin also proved a sharp lower bound for $D(G) + D\left(\overline{G}\right)$.

There is a simple method that often produces a product upper bound when a sum upper bound is known. It is well-known that $\sqrt{xy} \leq \frac{x+y}{2}$ with equality exactly when $x = y$, which justifies the following lemma.

Lemma 4. [14] *If $p(G) + p\left(\overline{G}\right) \leq f(n)$, then $p(G) \cdot p\left(\overline{G}\right) \leq \frac{(f(n))^2}{4}$, with equality exactly when $p(G) = p\left(\overline{G}\right) = \frac{f(n)}{2}$.*

Next we apply this to degeneracy.

Corollary 5. *For any graph G, $0 \leq D(G) \cdot D\left(\overline{G}\right) \leq \left(\frac{n-1}{2}\right)^2$. The lower bound is an equality exactly for $\left\{K_n, \overline{K}_n\right\}$. The upper bound is an equality exactly when the sum bound is attained and $D(G) = D\left(\overline{G}\right)$.*

Proof. The lower bound is obvious. Equality occurs exactly when one of the factors is 0, implying one is empty and the other is complete. The upper bound follows from Lemma 4. □

Aside from being interesting in its own right, degeneracy is also useful in proving the theorems on vertex coloring to follow.

2. Chromatic Number

Definition 6. The chromatic number of a graph, $\chi(G)$, is the smallest number of subsets into which the vertices of a graph can be partitioned so that no two vertices in the same subset are adjacent. A graph is **critically** k**-chromatic** (k**-critical** when the context is clear) when $\chi(G) = k$ and $\chi(H) < k$ for any proper subgraph $H \subset G$.

We will need the well-known fact that a k-critical graph G has $\delta(G) \geq k-1$.
Degeneracy can be used to bound the chromatic number.

Theorem 7. (**The Degeneracy Bound**) *For any graph G, $\chi(G) \leq 1 + D(G)$.*

Proof. Color the vertices using the reverse of a deletion sequence. Each vertex has degree at most equal to its degeneracy when colored. Coloring it uses at most one more color. Thus $\chi(G) \leq 1 + D(G)$. □

The bound $\chi(G) \leq 1 + D(G)$ was apparently first published by Erdős and Hajnal [18], who describe it as "well-known". They call the quantity $1 + D(G)$ the "coloring number". The same bound was also published by Szekeres and Wilf [33] and many others.
The sum upper bound of the original Nordhaus-Gaddum Theorem follows immediately from the Degeneracy Bound.

Corollary 8. [30] *For any graph G, $\chi(G) + \chi\left(\overline{G}\right) \leq n+1$.*

Proof. We have $\chi(G)+\chi\left(\overline{G}\right) \leq 1+D(G)+1+D\left(\overline{G}\right) \leq n-1+2 = n+1$. □

We would like to characterize the extremal decompositions for the Nordhaus-Gaddum Theorem. Note that if a 2-decomposition of K_n achieves $\chi(G) + \chi\left(\overline{G}\right) = n+1$, then we can easily construct a 2-decomposition of K_{n+1} with $\chi(H) + \chi\left(\overline{H}\right) = n+2$, by letting $H = G + K_1$. Similarly, we may be able to delete some vertex v of K_n so that $\chi(G-v) + \chi\left(\overline{G-v}\right) = n$. If this is impossible, we say that an extremal decomposition is fundamental.

Definition 9. A decomposition $\left\{G, \overline{G}\right\}$ of K_n with $\chi(G) + \chi\left(\overline{G}\right) = n+1$ so that deleting any vertex v of K_n reduced the chromatic number of both factors is called a **fundamental decomposition**.

To characterize the extremal decompositions, we also need Brooks' Theorem.

Theorem 10. (**Brooks' Theorem**) [11] *If G is connected, then $\chi(G) = 1 + \triangle(G)$ if and only if G is complete or an odd cycle.*

Short proofs of Brooks' Theorem are available, e.g. in [6] and [16].

Lemma 11. *A fundamental 2-decomposition* $\{G, \overline{G}\}$ *of* K_n *has* $\chi(G) + \chi(\overline{G}) = n + 1$ *if and only if it is* $\{K_1, K_1\}$ *or* $\{C_5, C_5\}$.

Proof. ($\Leftarrow$) It is easily seen that $\chi(K_1) + \chi(K_1) = 2$ and $\chi(C_5) + \chi(C_5) = 6$, so these decompositions satisfy the equation. They are fundamental since no vertex can be deleted from the first, and deleting a vertex from the second produces $\{P_4, P_4\}$, and $\chi(P_4) + \chi(P_4) = 4$.
($\Rightarrow$) Consider a fundamental 2-decomposition $\{G, \overline{G}\}$. Then both graphs are connected. Let $\chi(G) = r$, so that $\chi(\overline{G}) = n + 1 - r$. Then $\delta(G) \geq r - 1$ and $\delta(\overline{G}) \geq n + 1 - r$. Now for any vertex v, $n - 1 \leq \delta(G) + \delta(\overline{G}) \leq d_G(v) + d_{\overline{G}}(v) \leq n - 1$. Thus we have equalities, so G and $\overline{G}$ must be regular. Now by Brooks' Theorem, the only connected regular graphs achieving $\chi(G) = 1 + D(G)$ are cliques and odd cycles. The only such graphs whose complements are connected and also achieve the upper bound are are K_1 and C_5. Thus the fundamental 2-decompositions are as stated. $\square$

We can now describe all extremal 2-decompositions for the upper bound of the Nordhaus-Gaddum theorem.

Theorem 12. *A 2-decomposition* $\{G, \overline{G}\}$ *of* K_n *has* $\chi(G) + \chi(\overline{G}) = n + 1$ *if and only if the critical subgraphs are* $\{K_{p+1}, K_{n-p}\}$ *or* $\{C_5 + K_p, C_5 + K_{n-p-5}\}$.

Proof. ($\Leftarrow$) Both decompositions have order n and $\chi(G) + \chi(\overline{G}) = n + 1$.
($\Rightarrow$) Assume that we have an extremal 2-decomposition with critical subgraphs G and H. They must overlap on some nonempty set of vertices S_2. Let S_1 be the set of vertices only in G, and let S_3 be the set of vertices only in H. Let G_i be G restricted to S_i, and similarly for H. Let $n_i = |S_i|$.

We have $\chi(G) + \chi(H) \leq \chi(G_1) + \chi(G_2) + \chi(H_2) + \chi(H_3) \leq n_1 + n_2 + 1 + n_3 = n + 1$. Thus the decomposition of S_2 consists of spanning regular critical graphs, so it is one of those listed in Lemma 11.

If $G_2 = K_1$ and $H_2 = K_1$, then $G \subseteq K_{n_1+1}$ and $H \subseteq K_{n_3+1}$. These must be equalities, since complete graphs are critical. If $G_2 = C_5$ and $H_2 = C_5$, then $G \subseteq C_5 + K_{n_1}$ and $H \subseteq C_5 + K_{n_3}$. These must be equalities, since $C_5 + K_r$ is critical. Set $p = n_1$, and we are done. $\square$

H. J. Finck [20] determined an equivalent but inelegant characterization whose proof is about 3.5 pages long. Starr and Turner [32] determined the following alternative characterization whose proof is about 3 pages long.

Theorem 13. [32] *Let G and $\overline{G}$ be complementary graphs on n vertices. Then $\chi(G)+\chi\left(\overline{G}\right)=n+1$ if and only if $V(G)$ can be partitioned into three sets S, T, and $\{x\}$ such that $G[S]=K_{\chi(G)-1}$ and $G[T]=K_{\chi\left(\overline{G}\right)-1}$.*

This characterization does not make it clear what G and $\overline{G}$ are. However, it is a corollary to Theorem 12.

Having characterized the graphs for which $\chi(G)+\chi\left(\overline{G}\right)=n+1$, we may consider when $\chi(G)+\chi\left(\overline{G}\right)=n$. Restricted to regular graphs, this is not a difficult problem.

Proposition 14. *A 2-decomposition $\left\{G,\overline{G}\right\}$ of K_n with regular factors has $\chi(G)+\chi\left(\overline{G}\right)=n$ if and only if it is $\left\{C_7,\overline{C}_7\right\}$ or $\{C_4,2K_2\}$.*

Proof. We have $n=\chi(G)+\chi\left(\overline{G}\right)\leq 1+D(G)+1+D\left(\overline{G}\right)=n+1$, so exactly one of G or $\overline{G}$ achieves the Degeneracy Bound, say G. If G is connected, then by Brooks' Theorem, G is a complete graph or odd cycle. But the complement of a complete graph also achieves the upper bound. If $G=C_n$ with $n\geq 5$ odd, then $\chi\left(\overline{C}_n\right)=\frac{n+1}{2}$. But $\overline{C}_n$ is $n-3$-regular, so $\frac{n+1}{2}=n-3$ implies $n=7$.

If G is disconnected, then it is a union of r-regular components, at least one of which is a clique or an odd cycle. Consider starting with only this component and adding another component with order k. This increases $\chi(G)+\chi\left(\overline{G}\right)$ by at most $k-r$. Thus to get $\chi(G)+\chi\left(\overline{G}\right)=n$ we want $r=1$, so the new component is K_2, and no other component can be added. Thus only the 2-decomposition $\{C_4,2K_2\}$ works. $\square$

The product upper bound follows from Theorem 12 using Lemma 4. This bound is attained exactly when the sum bound is attained and $\chi(G)=\chi\left(\overline{G}\right)$.

Corollary 15. [30, 20] *For any graph G, $\chi(G)\cdot\chi\left(\overline{G}\right)\leq\left(\frac{n+1}{2}\right)^2$. The bound is attained exactly for $\left\{K_{\frac{n+1}{2}},K_{\frac{n+1}{2}}\right\}$ and $\left\{C_5+K_{\frac{n-5}{2}},C_5+K_{\frac{n-5}{2}}\right\}$.*

Finck's proof of the the extremal graphs is about 1.5 pages.

3. Fractional and Circular Chromatic Number

We can easily determine the extremal decompositions for some graph coloring parameters that are less than the chromatic number.

Definition 16. An k**-set coloring** assigns each vertex of a graph G a set of k colors so that adjacent vertices receive disjoint sets of colors. Let $\chi^k(G)$ be the minimum number of colors in an k-set coloring of G. The **fractional chromatic number** $\chi_f(G)$ of a graph G is $\chi_f(G) = \lim\limits_{k\to\infty} \frac{\chi^k(G)}{k}$.

Let k and d be positive integers with $k \geq 2d$. A (k,d)**-coloring** of a graph G on n vertices is a mapping $f : V(G) \to \{0, 1, ..., k-1\}$ such that $d \leq |f(u) - f(v)| \leq k - d$ for any $uv \in E(G)$. The **circular chromatic number** $\chi_c(G)$ of G is the minimum of $\frac{k}{d}$ for which there exists a (k,d)-coloring of G.

It is easily seen that $\max\left\{\omega(G), \frac{n}{\alpha(G)}\right\} \leq \chi_f(G) \leq \chi_c(G) \leq \chi(G)$. We have $\chi_f(C_5) = \frac{5}{2}$ using the 2-set coloring -12-34-51-23-45-, and also $\chi_c(C_5) = \frac{5}{2}$ [36]. Brown and Hoshino [12] proved the following inequality.

Corollary 17. *For any graph G, $\chi_f(G) + \chi_f\left(\overline{G}\right) \leq \chi_c(G) + \chi_c\left(\overline{G}\right) \leq n+1$. A 2-decomposition $\left\{G, \overline{G}\right\}$ of K_n has $\chi_f(G) + \chi_f\left(\overline{G}\right) = n+1$ or $\chi_c(G) + \chi_c\left(\overline{G}\right) = n+1$ if and only if the critical subgraphs are $\{K_{p+1}, K_{n-p}\}$.*

Proof. By the Nordhaus-Gaddum Theorem,

$$\chi_f(G) + \chi_f\left(\overline{G}\right) \leq \chi_c(G) + \chi_c\left(\overline{G}\right) \leq \chi(G) + \chi\left(\overline{G}\right) \leq n+1.$$

If $\chi_f(G) + \chi_f\left(\overline{G}\right) = n+1$, then $\chi(G) + \chi\left(\overline{G}\right) = n+1$, so we only need to check the decompositions that satisfy the latter inequality. By Theorem 12, these are $\{K_{p+1}, K_{n-p}\}$ or $\{C_5 + K_p, C_5 + K_{n-p-5}\}$. Now $\chi_f(K_{p+1}) + \chi_f(K_{n-p}) = n+1$, but $\chi_c(C_5 + K_p) + \chi_c(C_5 + K_{n-p-5}) = n$, since $\chi_c(C_5) = \frac{5}{2}$. $\square$

It is easy to prove a similar result for the clique number $\omega(G)$. We have $\omega(G) + \omega\left(\overline{G}\right) = n+1$ exactly when the critical subgraphs are $\{K_{p+1}, K_{n-p}\}$.

4. List Chromatic Number

Next we consider a Nordhaus-Gaddum theorem for list coloring. The techniques used for chromatic number also work here, with some modifications.

Definition 18. A **list coloring** of a graph begins with lists of length k assigned to each vertex and chooses a color from each list to obtain a proper vertex coloring. A graph G is k**-choosable** if any assignment of lists to the vertices permits a proper coloring. The **list chromatic number** $\chi_l(G)$, is the smallest k such that G is k-choosable.

We have the following basic bounds on list chromatic number.

Theorem 19. [19] *For any graph G, $\chi(G) \leq \chi_l(G) \leq 1 + D(G)$.*

The lower bound follows since the lists could be identical, and the upper bound has the same proof as Theorem 7. Brooks' Theorem extends to list coloring.

Theorem 20. (**Brooks' Theorem for list coloring**) *If G is connected, then $\chi_l(G) = 1 + \triangle(G)$ if and only if G is complete or an odd cycle.*

Borodin [9] and Erdős, Rubin, and Taylor [19] originally proved this as part of a more general result with a longer proof. It was also proved by Vizing [34]. See [35] for a fairly short proof.

Corollary 8 and Lemma 11 generalize to list coloring by replacing χ with χ_l in their statements and proofs.

Corollary 21. [19] *For any graph G, $\chi_l(G) + \chi_l\left(\overline{G}\right) \leq n + 1$.*

Proof. We have $\chi_l(G)+\chi_l\left(\overline{G}\right) \leq 1+D(G)+1+D\left(\overline{G}\right) \leq n-1+2 = n+1$. $\square$

The extremal decompositions for list coloring include a larger class of graphs. Let H be a graph and let $f(H)$ be the smallest k so that $\chi_l\left(H + \overline{K}_k\right) > n(H)$. This function is well-defined since $\chi_l(K_{r,r^r}) = r+1$. Dantas, Gravier, and Maffray [17] proved the following, phrased in terms of conditions on a graph, not in terms of decompositions. Their proof is about 5 pages.

Theorem 22. [17] *Let H be a graph and let $f(H)$ be the smallest k so that $\chi_l\left(H + \overline{K}_k\right) > n(H)$. A 2-decomposition $\left\{G, \overline{G}\right\}$ of K_n has $\chi_l(G) + \chi_l\left(\overline{G}\right) = n + 1$ if and only if the critical subgraphs are $\left\{K_{s+k}, H + \overline{K}_k\right\}$ or $\{C_5 + K_p, C_5 + K_{n-p-5}\}$.*

Proof. ($\Leftarrow$) We see $\chi_l(K_{s+k}) + \chi_l\left(H + \overline{K}_k\right) = s + k + n(H) + 1 = n + 1$ and $\chi_l(C_5 + K_p) + \chi_l(C_5 + K_{n-p-5}) = 3 + p + 3 + (n - p - 5) = n + 1$.
($\Rightarrow$) Assume that we have an extremal 2-decomposition with critical subgraphs G and H. They must overlap on some nonempty set of vertices S_2. Let S_1 be the set of vertices only in G, and let S_3 be the set of vertices only in H. Let G_i be G restricted to S_i, and similarly for H. Let $n_i = |S_i|$.

Now if G_2 and H_2 are not regular, we have $D(G)+D(H) < n_2-1+n_1+n_3 = n-1$, and $\chi_l(G) + \chi_l(H) < n + 1$. Now by Brooks' Theorem for list coloring, the only connected regular graphs achieving $\chi(G) = 1 + D(G)$ are cliques and odd cycles. The only such graphs whose complements also achieve the upper bound are are K_r and C_5. Thus the decomposition restricted to S_2 is $\left\{K_r, \overline{K}_r\right\}$ or $\{C_5, C_5\}$.

In the first case, say $G_2 = K_r$. Note that $\chi_l(H) \leq 1 + n_3$ and $\chi_l(G) \leq n_1 + n_2$. For the decomposition to be extremal, these must both be equalities. Thus $G = K_{n_1+n_2}$. Now every edge joining S_2 and S_3 must be in H, or else $\delta(H) < n_3$. Also,n_2 must be large enough that $\chi_l\left(H_3 + \overline{K}_{n_2}\right) > n_3$. Thus $n_2 \geq f(H_3)$.

In the second case, $\chi_l(G) \leq n_1 + 3$. For this to be an equality, every edge joining S_1 and S_2 must be in G, or else $\delta(G) < n_1 + 2$. Also, every edge in S_1 must be in G, since $\chi_l((K_{n_1} - e) + C_5) = n_1 + 2$ (this follows from results in [19] or [22]). The argument is similar for H. $\square$

This theorem implies Theorem 12 as a corollary.

The function $f(H)$ was further explored in [23], but has not been completely characterized. They showed that $f(K_n) = 1$ and $f(H) \geq n^2$ for $H \subset K_n$ with order n. This along with Lemma 4 implies the following.

Corollary 23. *For any graph G, $\chi_l(G) \cdot \chi_l\left(\overline{G}\right) \leq \left(\frac{n+1}{2}\right)^2$. The bound is attained exactly for $\left\{K_{\frac{n+1}{2}}, K_{\frac{n+1}{2}}\right\}$ and $\left\{C_5 + K_{\frac{n-5}{2}}, C_5 + K_{\frac{n-5}{2}}\right\}$.*

5. $L(2,1)$ Labeling

Another type of vertex coloring restricts the differences between colors of vertices at different distances.

Definition 24. An $L(2,1)$ **coloring** c of a graph G is an assignment of colors (nonnegative integers) to the vertices of G such that if u and w are adjacent vertices of G, then $|c(u) - c(w)| \geq 2$ while if $d(u,w) = 2$, then $|c(u) - c(w)| \geq 1$. Given an $L(2,1)$ coloring c of a graph G, the **c-span** of G is $\lambda_{2,1}(c) = \max\limits_{u,w \in G} |c(u) - c(w)|$. The **span** of G is $\lambda(G) = \min\{\lambda_{2,1}(c)\}$.

Griggs and Yeh [24] proved a basic upper bound.

Proposition 25. *For any graph G, $\lambda(G) \leq n + \chi(G) - 2$.*

It is not hard to show that equality holds exactly for complete multipartite graphs with order n (see [15]).

Balakrishnan and Deo [2] determined a Nordhaus-Gaddum theorem for span, including the bound $\lambda(G) + \lambda\left(\overline{G}\right) \leq 3n - 3$. However, this is not sharp.

Theorem 26. *For any graph G with order $n \geq 2$, $\lambda(G) + \lambda\left(\overline{G}\right) \leq 3n - 4$. The extremal 2-decompositions are $\{K_{1,n-1}, K_{n-1}\}$.*

Proof. By Proposition 25 and the Nordhaus-Gaddum Theorem,

$$\begin{aligned} \lambda(G) + \lambda\left(\overline{G}\right) \leq & n + \chi(G) - 2 + n + \chi\left(\overline{G}\right) - 2 \\ \leq & 2n - 4 + (n+1) \\ = & 3n - 3. \end{aligned}$$

For the bound to be an equality, both G and $\overline{G}$ must be complete multipartite graphs with order n. However, the complement of a complete multipartite graph is a disjoint union of complete graphs. This is only a complete multipartite graph when $n = 1$. Thus $\lambda(G) + \lambda\left(\overline{G}\right) \leq 3n - 4$ holds when $n \geq 2$.

For this to be an equality, one of the factors (say G) must be a complete multipartite graph with order n. Then $\overline{G}$ must have a component with order $n - 1$, so $\{K_{1,n-1}, K_{n-1}\}$ is the only extremal decomposition. □

The product upper bound follows from Lemma 4.

Corollary 27. *For any graph G with order $n \geq 2$, $\lambda(G) \cdot \lambda\left(\overline{G}\right) \leq \frac{(3n-4)^2}{4}$.*

This is only attained when $n = 4$.

6. Vertex Arboricity and Point Partition Number

Proper vertex coloring studies partitioning the vertex set of a graph into independent sets, which are 0-degenerate graphs. Thus it is a natural question to consider partitions of the vertices of a graph into sets that induce k-degenerate graphs.

Theorem 28. *Let G be a graph and $k_1, ..., k_t$ be nonnegative integers with $\sum k_i \geq D(G) - t + 1$. Then the vertices of G can be partitioned into sets $V_1, ..., V_t$ so that $D(G[V_i]) \leq k_i$.*

Proof. Let $k = D(G)$, and consider the reverse of a deletion sequence for G. We use induction on order n. The result is clear for the first vertex. Assume that for the first r vertices, there is a vertex partition with each set having $D(G[V_i]) \leq k_i$. The next vertex v added is adjacent to at most k existing vertices. If v had at least $k_i + 1$ neighbors in V_i for all i, $k \geq \sum(k_i + 1) =$

$\sum k_i + t$, a contradiction. Thus the Pigeonhole Principle says there is some (possibly empty) set V_i of vertices with $D(G[V_i]) < k_i$. Adding v to this set, we find $D(G[V_i \cup v]) \leq k_i$. Thus the result holds for G by induction. $\square$

In the case when $k_1 = ... = k_t$, the following definition is natural.

Definition 29. The **point partition number** $\rho_k(G)$ is the minimum number of sets into which the vertices of a graph G can be partitioned so that each set induces a k-degenerate graph. The **vertex arboricity** of G is $a(G) = \rho_1(G)$.

Point partition numbers were first introduced in 1970 by Lick and White [26] in the same paper that introduced k-degenerate graphs. It is immediate that $\chi(G) = \rho_0(G) \leq \rho_1(G) \leq \rho_2(G) \leq \ldots \leq \rho_k(G) \leq \ldots$.

There was a flurry of research on these numbers in the 1970s. Surveys of results on these numbers appear in [31] and [7]. The following generalization of the Degeneracy bound follows immediately from Theorem 28.

Corollary 30. [26] *For any graph G, $\rho_k(G) \leq 1 + \left\lfloor \frac{1}{k+1} D(G) \right\rfloor$.*

This bound is exact whenever $0 \leq D(G) \leq 2k+1$ since any graph with $k+1 \leq D(G) \leq 2k+1$ is not k-degenerate, but the upper bound is two. The corresponding bound for vertex arboricity was proved by Chartrand and Kronk [13].

Definition 31. A graph is **d-critical** with respect to ρ_k if $\rho_k(H) < \rho_k(G) = d$ for any proper subgraph $H \subset G$.

Corollary 30 implies a short proof of the following result from [26].

Corollary 32. [26] *If G has $\rho_k(G) = d$ and is critical with respect to point partition number, then $\delta(G) \geq (k+1)(d-1)$.*

Proof. Assume to the contrary that $d(v) \leq (k+1)(d-1) - 1$. Since G is critical, $G - v$ has a vertex partition inducing $d-1$ k-degenerate graphs. By the Pigeonhole Principle, v is adjacent to at most k vertices of one of them. But then $\rho_k(G) = d - 1$, a contradiction.

$\square$

There is a Nordhaus-Gaddum theorem for point partition number due to Lick and White [27], whose proof is about 1.5 pages. We present a much shorter proof.

Theorem 33. *For any graph G, $\rho_k(G) + \rho_k(\overline{G}) \leq \left\lfloor \frac{n+1+2k}{k+1} \right\rfloor$.*

Proof. By Corollary 30 and Theorem 3,

$$\begin{aligned}\rho_k(G) + \rho_k(\overline{G}) \leq & 1 + \left\lfloor \frac{1}{k+1} D(G) \right\rfloor + 1 + \left\lfloor \frac{1}{k+1} D(\overline{G}) \right\rfloor \\ \leq & 2 + \left\lfloor \frac{1}{k+1} \left(D(G) + D(\overline{G}) \right) \right\rfloor \\ \leq & 2 + \left\lfloor \frac{n-1}{k+1} \right\rfloor \\ = & \left\lfloor \frac{n+1+2k}{k+1} \right\rfloor .\end{aligned}$$

□

The corresponding result for vertex arboricity is due to Mitchem [28], where it has a 1-page proof. Mitchem also showed that this bound is sharp.

There is also a generalization of Brooks' Theorem for point partition number, due to Mitchem [29]. The corresponding result for vertex arboricity is due to Kronk and Mitchem [25], and a short proof using degeneracy is in [5].

Theorem 34. [29] *If G is connected, then $\rho_k(G) = 1 + \frac{1}{k+1} \triangle(G)$ if and only if G is*

1. *$K_{t(k+1)+1}$*
2. *$k+1$-regular*
3. *an odd cycle, when $k = 0$.*

As before, we say that a decomposition is fundamental if deleting any vertex reduces the parameter (in this case, point partition number) in both factors.

Lemma 35. *A fundamental 2-decomposition $\left\{G, \overline{G}\right\}$ of K_n has $\rho_k(G) + \rho_k(\overline{G}) = \frac{n+1+2k}{k+1}$ if and only if it is one of the following.*

1. *$\{K_1, K_1\}$ for all k*
2. *$\left\{G, \overline{G}\right\}$ where G is $k+1$-regular of order $2k+3$ with $k \geq 1$ odd*
3. *$\{C_5, C_5\}$ for $k = 0$*

The only other decomposition into spanning regular graphs that attains the bound is $\left\{K_{t(k+1)+1}, \overline{K}_{t(k+1)+1}\right\}$.

Proof. (⇐) Clearly $\rho_k(K_1)+\rho_k(K_1)=2$, and if G is $k+1$-regular of order $2k+3$, $\rho_k(G)+\rho_k\left(\overline{G}\right)=4=\frac{2k+3+1+2k}{k+1}$, so these decompositions satisfy the equation. They are fundamental since no vertex can be deleted from the first, and deleting a vertex from the second reduces the sum to 2.

(⇒) Consider a fundamental 2-decomposition $\left\{G,\overline{G}\right\}$. Then both graphs are connected. Let $\rho_k(G)=r$, so that $\rho_k\left(\overline{G}\right)=\frac{n+1+2k}{k+1}-r$. Then $\delta(G)\geq(k+1)(r-1)$ and $\delta\left(\overline{G}\right)\geq(k+1)\left(\frac{n+1+2k}{k+1}-r-1\right)=n+1+2k-(k+1)(r+1)$. Now for any vertex v, $n-1=(k+1)(r-1)+(n+1+2k-(k+1)(r+1))\leq\delta(G)+\delta\left(\overline{G}\right)\leq d_G(v)+d_{\overline{G}}(v)\leq n-1$. Thus we have equalities, so G and $\overline{G}$ must be regular.

Now Theorem 34 gives the only connected regular graphs achieving $\rho_k(G)=1+\frac{1}{k+1}\triangle(G)$. The only such graphs whose complements are connected and also achieve the upper bound are are K_1, a $k+1$-regular graph of order $2k+3$ with $k\geq1$ odd, and C_5 for $k=0$. Thus the fundamental 2-decompositions are as stated.

If the decomposition need not be fundamental, one of the two factors must still be connected and Brooks' Theorem applies. The only possibility remaining is $\left\{K_{t(k+1)+1},\overline{K}_{t(k+1)+1}\right\}$. □

We can now describe all extremal 2-decompositions for the upper bound of the Nordhaus-Gaddum theorem for point partition number.

Theorem 36. *A 2-decomposition $\left\{G,\overline{G}\right\}$ of K_n has $\rho_k(G)+\rho_k\left(\overline{G}\right)=\frac{n+1+2k}{k+1}$ if and only if the critical subgraphs are one of the following.*

1. $\left\{G_2+K_{(k+1)p},\overline{G_2}+K_{(k+1)q}\right\}$ *where G_2 is $k+1$-regular of order $2k+3$ with $k\geq1$ odd*
2. $\left\{K_{(k+1)p+1},K_{(k+1)q+1}\right\}$
3. $\left\{K_{(k+1)r+1},H_3+\overline{K}_{k+1-d}\right\}$, *where $n(H_3)=k+1$, $\delta(H_3)=d$, and H_3 has no adjacent vertices with degree more than d*
4. $\{C_5+K_p,C_5+K_{n-p-5}\}$ *for $k=0$*

Proof. (⇐) In case 1, $n=(k+1)p+(k+1)q+2k+3=(k+1)(p+q+2)+1$ and $\sum\rho_k=p+q+4$. In case 2, $n=(k+1)p+(k+1)q+1=(k+1)(p+q)+1$ and $\sum\rho_k=p+q+2$. In case 3, $n=(k+1)r+1+k+1=(k+1)(r+1)+1$ and $\sum\rho_k=r+3$. Case 4 is contained in Theorem 12. Thus in each case, $\rho_k(G)+\rho_k\left(\overline{G}\right)=\frac{n+1+2k}{k+1}$.
(⇒) Assume that we have an extremal 2-decomposition with critical subgraphs G and H. They must overlap on some nonempty set of vertices S_2.

Let S_1 be the set of vertices only in G, and let S_3 be the set of vertices only in H. Let G_i be G restricted to S_i, and similarly for H. Let $n_i = |S_i|$.

Since $\rho_k(G) \leq \rho_k(G_1 + G_2) \leq \rho_k(G_1) + \rho_k(G_2)$, we have $\rho_k(G) + \rho_k(H) \leq \rho_k(G_1) + \rho_k(G_2) + \rho_k(H_2) + \rho_k(H_3) \leq \frac{n_1}{k+1} + \frac{n_2+1+2k}{k+1} + \frac{n_3}{k+1} = \frac{n+1+2k}{k+1}$. Thus n_1 and n_3 are divisible by $k+1$, and the decomposition of S_2 consists of spanning regular graphs, so it is one of those listed in Lemma 35.

If G_2 and H_2 are $k+1$-regular of order $2k+3$ with $k \geq 1$ odd then $G \subseteq G_2 + K_{(k+1)p}$. Since G is critical, $\delta(G) \geq (k+1)(p+1)$. Thus all edges incident with vertices of S_2 are in G. Now $\rho_k\left(G_2 + \left(K_{(k+1)p} - e\right)\right) < \rho_k\left(G_2 + K_{(k+1)p}\right)$, so $G = G_2 + K_{(k+1)p}$. Similarly, $H = \overline{G}_2 + K_{(k+1)q}$.

If $G_2 = K_{t(k+1)+1}$ and $H_2 = \overline{K}_{t(k+1)+1}$, then we must have $G = K_{(k+1)(t+p)+1}$, which is critical. Also, $H \subseteq \overline{K}_{(k+1)t+1} + K_{(k+1)q}$, and since H is critical, $\delta(H) \geq (k+1)q$. Thus all edges incident with vertices of S_2 are in H. When $q \geq 2$, $\rho_k\left(\overline{K}_{(k+1)t+1} + \left(K_{(k+1)q} - e\right)\right) < \rho_k\left(\overline{K}_{(k+1)t+1} + K_{(k+1)q}\right)$. Thus $K_{(k+1)q+1}$ is the only critical subgraph, so $t = 0$.

When $q = 1$, we must have $\delta(H) = k+1$. Thus for some $d \geq 0$, $\delta(H_3) = d$, H_3 has no adjacent vertices with degree more than d, and $H = H_3 + \overline{K}_{k+1-d}$ is critical.

The case when $k = 0$, $G_2 = C_5$ and $H_2 = C_5$ was proved in Theorem 12. □

When $k = 0$, this just yields Theorem 12. Restricted to vertex arboricity, this theorem says the following.

Corollary 37. *A 2-decomposition $\left\{G, \overline{G}\right\}$ of K_n has $a(G) + a\left(\overline{G}\right) = \frac{n+3}{2}$ if and only if the critical subgraphs are one of the following.*

1. $\{K_{2p+1}, K_{2q+1}\}$
2. $\{C_5 + K_{2p}, C_5 + K_{2q-4}\}$
3. $\{C_4, K_{n-2}\}$, *$n \geq 5$ odd*

The product upper bound for point partition number follows from Lemma 4.

Corollary 38. *A 2-decomposition $\left\{G, \overline{G}\right\}$ of K_n has $\rho_k(G) \cdot \rho_k\left(\overline{G}\right) = \frac{1}{4}\left(\frac{n+1+2k}{k+1}\right)^2$ if and only if the critical subgraphs are one of the following.*

1. $\left\{G_2 + K_{(k+1)p}, \overline{G_2} + K_{(k+1)p}\right\}$ *where G_2 is $k+1$-regular of order $2k+3$ with $k \geq 1$ odd*
2. $\left\{K_{(k+1)p+1}, K_{(k+1)p+1}\right\}$
3. $\left\{K_{k+2}, H_3 + \overline{K}_{k+1-d}\right\}$, *where $n(H_3) = k+1$, $\delta(H_3) = d$, and H_3 has no adjacent vertices with degree more than d*

4. $\left\{C_5 + K_{\frac{n-5}{2}}, C_5 + K_{\frac{n-5}{2}}\right\}$ *for* $k = 0$

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