Nordhaus-Gaddum Results for Genus

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Abstract

We consider upper and lower bounds for $\gamma(G)+\gamma\left(\overline{G}\right)$, the sum of the genus of a graph and its complement. For the lower bound, we show $\gamma(G)+\gamma\left(\overline{G}\right) \geq \left\lceil \frac{1}{12}\left(n^2-13n+24\right)\right\rceil$. Furthermore, we construct an infinite family of graphs attaining this bound along with several other isolated examples. We provide a construction to show that $\gamma(G)+\gamma\left(\overline{G}\right)$ can be at least as large as $\frac{1}{48}\left(5n^2-52n+144\right)$, and determine sharp upper bounds for a few small orders. Some asymptotic results are considered.

Keywords: Nordhaus-Gaddum, genus, voltage graph, decomposition

1. Introduction

One common way to study a graph parameter $p(G)$ is to examine the sum $p(G)+p\left(\overline{G}\right)$ and product $p(G)\cdot p\left(\overline{G}\right)$. A theorem providing sharp upper and/or lower bounds for this sum and product is known as a theorem of the Nordhaus-Gaddum class. Of the four possible bounds, the sum upper bound has attracted the most attention. We will examine both sum bounds for genus. (See [8] for background on genus; see [7] for basic definitions and notation.)

It is convenient to consider a graph and its complement as a decomposition of a complete graph. This makes it possible to generalize the problem to more than two factors. This problem has been previously considered in [6].

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Definition 1. A *k-decomposition* of a graph G is a decomposition of G into $k \geq 2$ factors. For a graph parameter p, let $p\,(k;G)$ denote the maximum of $\sum_{i=1}^{k} p\,(G_i)$ over all k-decompositions of G.

Hence we use the notation $\gamma\,(k;K_n)$ to denote the maximum of $\sum_{i=1}^{k} \gamma\,(G_i)$ over all k-decompositions of K_n. (Since G_i may be disconnected, we allow embedding on disconnected surfaces.) We will tend to describe a k-decomposition as $\{H_1, ..., H_k\}$, where each H_i is a critical factor of the decomposition.

It is trivial that $\prod_{i=1}^{k} \gamma\,(G_i) \geq 0$ and this bound is sharp for all orders. The decompositions that attain equality are exactly those in which at least one of the factors is planar. Little is currently known about an upper bound for $\prod_{i=1}^{k} \gamma\,(G_i)$. The rest of this project focuses on upper and lower bounds for $\sum_{i=1}^{k} \gamma\,(G_i)$.

2. The sum upper bound

For genus, it is obvious that $\gamma\,(k;K_n) = 0$ for $1 \leq n \leq 4$. Is it easily checked that $\gamma\,(k;K_n) = 1$ for $5 \leq n \leq 7$ since the complements of $K_5 + \overline{K}_2$ $(K_2 \vee \overline{K}_5)$, $K_{3,3} + K_1$ $(K_1 \vee 2K_3)$, and of the graphs formed by subdividing edges of K_5 and $K_{3,3}$ up to order 7 are planar. (Note that $+$ indicates disjoint union, and $\vee$ indicates join.)

Proposition 2. *We have $\gamma\,(k;K_8) = 2$.*

Proof. The decompositions $\{K_8\}$, $\{K_{3,3}, K_2 \vee 2K_3\}$, $\left\{K_5, K_3 \vee \overline{K}_5\right\}$ all show $\gamma\,(k;K_8) \geq 2$. Now Duke and Haggard [5] characterized all minimal graphs with order 8 and genus two, namely $K_8 - K_3$, $K_8 - (K_2 + P_3)$, $K_8 - K_{2,3}$. The complements of these graphs are all planar.

If there are three nonplanar graphs which decompose K_8, their sizes sum to 28. It is easily checked that the triples of graphs $\{K_{3,3}, K_{3,3}, K_{3,3}\}$, $\{K_{3,3}, K_{3,3}, K_{3,3} \sim e\}$, $\{K_{3,3}, K_{3,3}, K_5\}$ (where $K_{3,3} \sim e$ is a subdivistion of $K_{3,3}$) are the only possibilties and none of them decompose K_8 since some vertex would have degree at least 3 in each of the factors. Hence $\gamma\,(k;K_8) = 2$. □

Certainly $\gamma\,(k;K_n) \geq \gamma\,(K_n)$. We show that no decomposition into genus one graphs can produce $\gamma\,(k;K_n) > \gamma\,(K_n)$.

Lemma 3. *K_{10} does not decompose into five copies of $K_{3,3}$.*

Proof. Suppose that such a decomposition D exists. Let D' be the decomposition induced by the six vertices of the factor $G_1 = K_{3,3}$. Now no other factor of D' can contain two independent edges, so the other four factors of D' must be two P_3's and two K_2's. Now each vertex of D is contained in three factors. Let v be a vertex of D' that contains the central vertex of one of the P_3's. There must be another factor containing v in D, and it cannot be any of the existing five since the edges to the parts of them contained in D' have already been used. But this requires six factors, a contradiction. $\square$

Theorem 4. *K_n decomposes into at most $\gamma(K_n)$ nonplanar graphs.*

Proof. We have already seen that this is true for $1 \leq n \leq 8$. For $n = 9$, $\gamma(K_9) = 3$ and $m(K_9) = 36$ (where $m(G)$ is the number of edges of G), so the only possibility to check is four $K_{3,3}$'s. But then some vertex would have degree 9 in the decomposition. For $n = 10$, $\gamma(K_{10}) = 4$ and $m(K_{10}) = 45$, so the only possibility is five $K_{3,3}$'s, which is impossible by the lemma.

For $n = 11$, $\gamma(K_{11}) = 5$ and $m(K_{11}) = 55$, so any counterexample must have five $K_{3,3}$'s and one other nontoroidal graph. But then some vertex would have degree at least 12 in the decomposition.

For $n = 12$, $\gamma(K_{12}) = 6$ and $m(K_{11}) = 66$, so any counterexample must have seven topological K_5's or $K_{3,3}$'s. Let x be the number of TK_5's and y be the number of $TK_{3,3}$'s. Then $x + y = 7$. Now at most three vertices of degree at least 3 occur at a given vertex of the decomposition, so $5x + 6y \leq 3 \cdot 12 = 36$. Combining these two conditions implies $x \geq 6$. But then three vertices of degree four occur at some vertex of the decomposition, which is impossible.

Since the minimum size of a nonplanar graph is 9, K_n can decompose into at most $\frac{1}{9}\binom{n}{2} = \frac{n(n-1)}{18}$ nonplanar graphs. We show that for $n \geq 13$, $\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \geq \left\lfloor \frac{n(n-1)}{18} \right\rfloor$. This will occur when $\frac{(n-3)(n-4)}{12} + \frac{11}{12} \geq \frac{n(n-1)}{18}$. This implies that $3(n^2 - 7n + 23) \geq 2(n^2 - n)$, so $n^2 - 19n + 69 \geq 0$. Then $n \geq \frac{19+\sqrt{19^2-4\cdot 69}}{2} > 14$. It is easily checked that the inequality also holds for $13 \leq n \leq 14$. $\square$

A lower bound on the size of graphs with genus $k \geq 1$ would help to address this problem. If is easily seen that for $n \geq 6$, $m = n + 3$ is the smallest size of a critical nonplanar graph. The size of the smallest order 8 genus 2 graph, $K_8 - K_{2,3}$, is 22.

There is a construction for which $\gamma(k; K_n) > \gamma(K_n)$ for large enough n.

Theorem 5. *For $n \geq 5$, we have*

$$\frac{5n^2 - 52n + 143}{48} \leq \gamma(2; K_n) \leq 2\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

Proof. This is easily checked for $n = 5$. For n even, consider the decomposition is $\left\{K_{\frac{n}{2},\frac{n}{2}}, 2K_{\frac{n}{2}}\right\}$. This gives

$$\sum_{i=1}^{2} \gamma(G_i) = \left\lceil \frac{1}{4}\left(\frac{n}{2} - 2\right)^2 \right\rceil + 2\left\lceil \frac{1}{12}\left(\frac{n}{2} - 3\right)\left(\frac{n}{2} - 4\right) \right\rceil$$

$$= \left\lceil \frac{(n-4)^2}{16} \right\rceil + 2\left\lceil \frac{(n-6)(n-8)}{48} \right\rceil \geq \frac{5n^2 - 52n + 144}{48}.$$

If n is odd, consider the decomposition $\left\{K_{\frac{n+1}{2},\frac{n-1}{2}}, K_{\frac{n+1}{2}} + K_{\frac{n-1}{2}}\right\}$. This gives

$$\sum_{i=1}^{2} \gamma(G_i) = \left\lceil \frac{1}{4}\left(\frac{n+1}{2} - 2\right)\left(\frac{n-1}{2} - 2\right) \right\rceil$$

$$+ \left\lceil \frac{1}{12}\left(\frac{n+1}{2} - 3\right)\left(\frac{n+1}{2} - 4\right) \right\rceil + \left\lceil \frac{1}{12}\left(\frac{n-1}{2} - 3\right)\left(\frac{n-1}{2} - 4\right) \right\rceil$$

$$= \left\lceil \frac{1}{16}(n-3)(n-5) \right\rceil + \left\lceil \frac{1}{48}(n-5)(n-7) \right\rceil + \left\lceil \frac{1}{48}(n-7)(n-9) \right\rceil$$

$$\geq \frac{5n^2 - 52n + 143}{48}.$$

For the upper bound, note that each factor in the 2-decomposition has genus $\gamma(G_i) \leq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$. □

Since $\frac{5}{48} > \frac{4}{48} = \frac{1}{12}$, this construction is superior to $\gamma(K_n)$ for large enough n. It is easily checked that the smallest value for which it is better is $n = 18$. The upper bound seems unlikely to be very good.

For larger numbers of factors, we can increase the lower bound. Note that $\gamma\left(K_{\frac{n}{2},\frac{n}{2}}\right) \approx \frac{n^2}{16}$. The remaining edges form two cliques of order $\frac{n}{2}$, so we can iterate this construction. If k factors are available, then

$$\sum_{i=1}^{k} \gamma(G_i) \approx \frac{1}{16}n^2 + \frac{2}{16}\left(\frac{n}{2}\right)^2 + ... + \frac{2^{k-2}}{16}\left(\frac{n}{2^{k-2}}\right)^2 + \frac{2^{k-1}}{12}\left(\frac{n}{2^{k-1}}\right)^2$$

$$= \frac{n^2}{16}\left(2 - \frac{1}{2^{k-2}} + \frac{4}{3}\frac{1}{2^{k-1}}\right) = \frac{n^2}{16}\left(2 - \frac{1}{3 \cdot 2^{k-2}}\right)$$

Thus for this construction, $\lim_{k\to\infty} \sum \gamma\left(G_i\right) \approx \frac{n^2}{8}$.

3. The sum lower bound

3.1. Proving the bound

For 2-decompositions, we have the following result.

Lemma 6. *Let $\left\{G, \overline{G}\right\}$ be a 2-decomposition of order n. Let the factors have h K_2-components together and j components with orders $n_i \geq 3$ together. Then*

$$\gamma\left(G\right) + \gamma\left(\overline{G}\right) \geq j + \frac{1}{6}\left(\binom{n}{2} - h\right) - \frac{1}{2}\sum_{i=1}^{j} n_i.$$

and this is an equality when each of the j components is a triangulation.

Proof. For $1 \leq i \leq j$, $\gamma_i \geq 1 + \frac{m_i}{6} - \frac{n_i}{2}$ ([8] Corollary 6-14). Now $\sum_{i=1}^{j} m_i + h = \binom{n}{2}$. Hence

$$\gamma\left(G\right) + \gamma\left(\overline{G}\right) = \sum_{i=1}^{j}\gamma_i \geq \sum_{i=1}^{j}\left(1 + \frac{m_i}{6} - \frac{n_i}{2}\right) = j + \frac{1}{6}\left(\binom{n}{2} - h\right) - \frac{1}{2}\sum_{i=1}^{j} n_i.$$

□

We would like to know how small this can be over all possible 2-decompositions.

Lemma 7. *For $n \geq 4$, there is a 2-decomposition that minimizes $\gamma\left(G\right) + \gamma\left(\overline{G}\right)$* for which both factors are connected.

Proof. Let $\left\{G, \overline{G}\right\}$ be a 2-decomposition that minimizes $\gamma\left(G\right) + \gamma\left(\overline{G}\right)$ and suppose G is connected and $\overline{G}$ is not. Now adding an edge between components of a disconnected graph does not increase its genus, and removing an edge from a graph cannot increase its genus. Hence we seek to move edges from G to $\overline{G}$ until the latter is also connected. We show that it is possible to do so without disconnecting G. This disconnection can only occur if the edge e that is moved is a bridge. Now G can contain at most $n-1$ bridges.

Suppose the smallest component of $\overline{G}$ has order $2 \leq r \leq \frac{n}{2}$. Since $r\left(n-r\right)$ is maximized at $r = \frac{n}{2}$ and decreases as r decreases, there are $r\left(n-r\right) > n-1$

edges in G joining that component to the rest of $\overline{G}$. These cannot all be bridges in G, so one of them can be shifted to $\overline{G}$ to unite two of its components without disconnecting G. If $\overline{G}$ has an isolated component, then G contains the $n-1$ edges incident with it, and these are all bridges exactly when $\overline{G} = K_{n-1} + K_1$ and $G = K_{1,n-1}$. But then for $n \geq 4$, G and $\overline{G}$ can swap edges so that both become connected without changing $\gamma(G) + \gamma\left(\overline{G}\right)$. $\square$

Theorem 8. *For $n \geq 3$, any 2-decomposition $\left\{G, \overline{G}\right\}$ has*

$$\gamma(G) + \gamma\left(\overline{G}\right) \geq \left\lceil \frac{1}{12}\left(n^2 - 13n + 24\right) \right\rceil .$$

Proof. This is easily checked for $n = 3$. By the lemma, among those decompositions that achieve the minimum for $n \geq 4$ there is one which has both factors connected. In Lemma 6, this yields $j = 2$, $h = 0$, and $\sum_{i=1}^{j} n_i = 2n$. Thus

$$\gamma(G) + \gamma\left(\overline{G}\right) \geq 2 + \frac{1}{6}\binom{n}{2} - \frac{1}{2}(2n) = \frac{1}{12}\left(n^2 - 13n + 24\right).$$

$\square$

Note that this is only a linear term away from $\gamma(K_n) = \left\lceil \frac{1}{12}\left(n^2 - 7n + 12\right) \right\rceil$, which says that increasing the number of factors does not decrease the sum of genera much.

3.2. Achieving the Bound

We now consider when this bound is sharp. The thickness of a graph is the smallest number of planar factors that decompose a graph. It is known [4] that

$$\theta(K_n) = \begin{cases} \left\lfloor \frac{n+7}{6} \right\rfloor & n \neq 9, 10 \\ 3 & n = 9, 10 \end{cases} .$$

In particular, $\theta(K_8) = 2$. This can be seen by noting that the graph $K_3 \Box K_3 - v$ is planar and self-complementary. Hence the minimum of $\gamma(G) + \gamma\left(\overline{G}\right)$ is 0 for $1 \leq n \leq 8$. However, since $\theta(K_9) = 3$, $\gamma(G) + \gamma\left(\overline{G}\right) \geq 1$ for $n \geq 9$. For $n = 9$, $\{K_{3,3,3}, 3K_3\}$ achieves $\gamma(G) + \gamma\left(\overline{G}\right) = 1$. For $n = 10$, $\{K_{2,2,2,2} + 2K_1, 4K_2 \vee K_2\}$ achieves $\gamma(G) + \gamma\left(\overline{G}\right) = 1$.

For $n = 12$, the bound is 1 and achieving it requires a maximal planar graph whose complement is maximal toroidal. White and Anderson [1] state that it is unknown whether this is possible. The complement of the icosahedron is nontoroidal, and there are more than 7000 other maximal planar graphs with order 12. However, the toroidal thickness $\theta_1(K_{12}) = 2$ [3], meaning that K_{12} can be decomposed into two toroidal graphs. Hence the minimum of $\gamma(G) + \gamma(\overline{G})$ is either 1 or 2 for $n = 12$.

Beineke [3] has shown that $\theta_2(K_n) = \left\lfloor \frac{n+3}{6} \right\rfloor$, which is the minimum number of subgraphs, each embeddable on S_2, whose union is K_n. For $n = 14$, we have $\theta_2(K_{14}) = 2$, so for a decomposition achieving this bound, $\gamma(G) + \gamma(\overline{G}) = 4$.

The theory of current and voltage graphs can be used to construct decompositions that achieve the bound. See chapter 10 of [8] for details regarding the use of voltage graphs. We now describe an infinite class of examples that achieve the bound.

Theorem 9. *The bound of Theorem 8 is attained for $n = 12s + 11$.*

Proof. The (index one) Ringel/Youngs current graphs for K_{12s+7}, $s \geq 0$, satisfy the Kirkoff Current Law in $\mathbb{Z}_\infty$ (see, for example, Figure 9-8 in [8]), so the corresponding (dual) voltage graphs satisfy the Kirkoff Voltage Law in $\mathbb{Z}_\infty$. They have $n = 1$, $m = 6s + 3$, $r = r_3 = 4s + 2$, with $\Gamma = \mathbb{Z}_{12s+7}$, and $\Delta = \{1, 2, ..., 6s + 3\}$.

Now set $\Gamma = \mathbb{Z}_{12s+9}$, with the same Δ. Since $\gcd(12s + 9, 6s + 4) = 1$, $G_{\overline{\Delta}}(\Gamma) = C_{12s+9}$. The covering graph for the above voltage graph embedding is $G_\Delta(\Gamma) = K_{12s+9} - C_{12s+9}$. The covering embedding has $n = 12s + 9$, $m = (6s + 3)(12s + 9)$, and $r = r_3 = (4s + 2)(12s + 9)$, so the genus of this embedding is $k = 1 + \frac{1}{2}(m - n - r) = 1 + \frac{1}{2}(12s + 9)(2s) = 12s^2 + 9s + 1$.

Finally, consider the 2-decomposition $\left\{G_\Delta(\Gamma) + K_2, C_{12s+9} \vee \overline{K}_2\right\}$, in which the double wheel $C_{12s+9} \vee \overline{K}_2$ is maximal planar. Then

$$\gamma(G) + \gamma(\overline{G}) = 12s^2 + 9s + 1 \geq \left\lceil \frac{(12s + 11)(12s - 2) + 24}{12} \right\rceil = 12s^2 + 9s + 1$$

so the bound is achieved for $n = 12s + 11$. □

For example, consider $n = 11$, with 2-decomposition $\left\{C_9 \vee 2K_1, \overline{C}_9 + K_2\right\}$. The voltage graph in the Figure below, using voltage group $\Gamma = \mathbb{Z}_9$, as generated by $\Delta = \{1, 2, 3\}$, so that the Cayley graph $G_\Delta(\Gamma) = K_9 - C_9$ shows that $\gamma(K_9 - C_9) = 1$. Thus $\gamma(G) + \gamma(\overline{G}) = 1$. Identification of opposite edges of the square embeds the one-vertex voltage graph into the torus, with

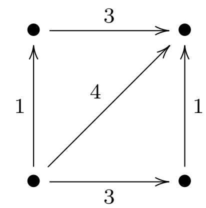

Figure 1: Voltage graph for $n = 11$.

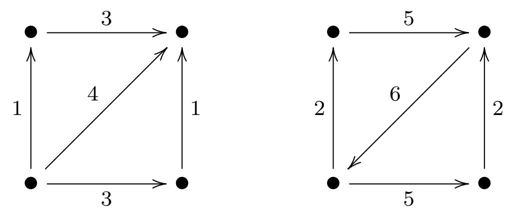

Figure 2: Voltage graphs for $n = 13$.

clockwise cycle permutation of edge labels $(1, 4, 3, 8, 5, 6)$. This lifts so that each of the nine covering vertices has this same clockwise cyclic ordering of edge labels on its incident edges. The "bottom" triangle $(3, 1, -4)$ given by edge labels lifts to triangle $(0, 3, 4)$ and its eight translates via $\mathbb{Z}_9$ given by vertex labels. Similarly, the "top" triangle below produces nine more triangles above. Thus the covering embedding is trianglular and gives the genus of the graph.

The bound is also achieved for orders 13, 25, 37, and 49.

For $n = 13$, as shown in [1], consider $\Gamma = \mathbb{Z}_{13}$ and $\Delta = \{1, 3, 4\}$, so $\overline{\Delta} = \{2, 5, 6\}$. Then both $G_\Delta(\Gamma)$ and $G_{\overline{\Delta}}(\Gamma)$ are toroidal. See Figure 2 for the voltage graphs used. Thus $\gamma(G) + \gamma\left(\overline{G}\right) = 2$.

For $n = 25$, consider $\Gamma = \mathbb{Z}_{25}$, $\Delta = \{2, 7, 9\}$, and $\overline{\Delta} = \{1, 3, 4, 5, 6, 8, 10, 11, 12\}$. Then the voltage graph embeddings of Figure 3 imply $\gamma(G) + \gamma\left(\overline{G}\right) = 1 + 26 = 27$.

For $n = 37$, consider the two arrays below, which we will call complementary magic squares (in rows and columns). Note also that if $\Delta = \{2, 4, 5, 7, 8, 9, 10, 12, 15\}$ for the first array, then $\overline{\Delta} = \{1, 3, 6, 11, 13, 14, 16, 17, 18\}$ for the second. We use the two arrays to give KVL triangles to find triangu-

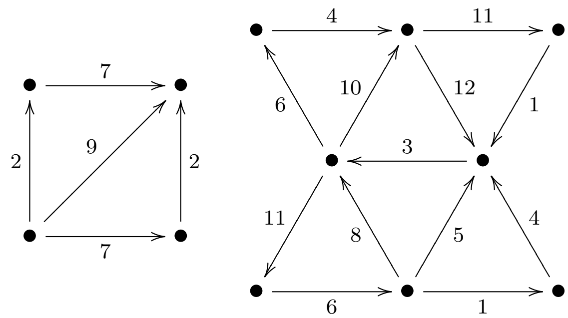

Figure 3: Voltage graphs for $n = 25$.

lar embeddings for $G_\Delta(\mathbb{Z}_{37})$ and for $G_{\overline{\Delta}}(\mathbb{Z}_{37}) = \overline{G_\Delta(\mathbb{Z}_{37})}$ respectively; each has genus 38. Thus the pair of embeddings satisfies $\gamma(G) + \gamma\left(\overline{G}\right) = 76$. We show the two voltage graph embeddings in Figure 4.

15	-5	-10
-8	-4	12
-7	9	-2

3	16	18
11	-17	6
-14	1	13

For $n = 49$, consider $\Gamma = \mathbb{Z}_{49}$, $\Delta = \{15, 16, 18\}$, and $\overline{\Delta} = \{1, ..., 14, 17, 19, ..., 24\}$. Then the voltage graph embeddings in Figure 5 imply $\gamma(G) + \gamma\left(\overline{G}\right) = 1 + 148 = 149$.

Note that the previous four values all satisfy $n = 12s + 1$, which suggests that they may be generalized to an infinite family.

Archdeacon and Grable [2] have shown that for a random graph G with edge probability $p = \frac{1}{2}$ and $\epsilon > 0$, $(1-\epsilon)\frac{n^2}{24} \leq \gamma(G) \leq (1+\epsilon)\frac{n^2}{24}$. Since the complement of a random graph is a random graph, for sufficiently large order, there is a 2-decomposition satisfying $\gamma(G) + \gamma\left(\overline{G}\right) \leq (1+\epsilon)\frac{n^2}{12}$. Hence the bound in Theorem 8 is nearly best possible.

Based on these constructions, we offer the following conjecture.

Conjecture 10. *For all $n \geq 11$, the lower bound $\left\lceil \frac{1}{12}(n^2 - 13n + 24) \right\rceil$ of $\gamma(G) + \gamma\left(\overline{G}\right)$ is attained by some graph G.*

Acknowledgement. *Thanks to Dan Archdeacon for his helpful suggestions.*

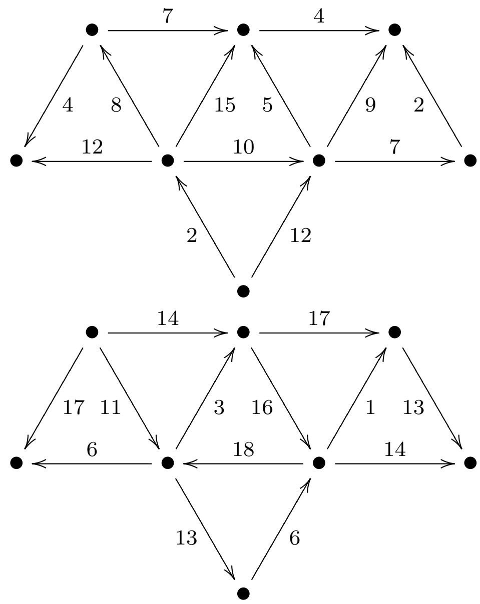

Figure 4: Voltage graphs for $n = 37$.

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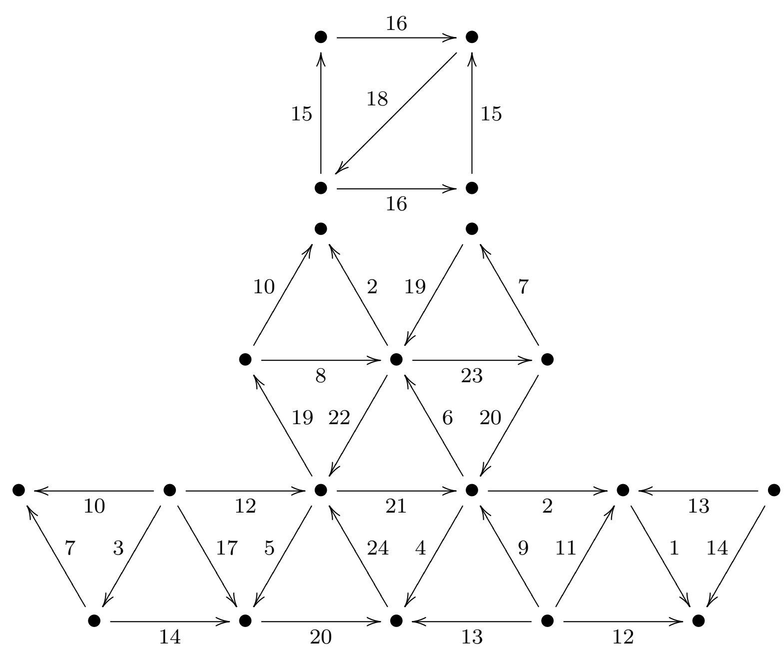

Figure 5: Voltage graphs for $n = 49$.

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