Nordhaus-Gaddum Theorems for Multifactor Decompositions

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Abstract

A Nordhaus-Gaddum theorem states bounds on $p(G)+p(\overline{G})$ and $p(G)\cdot p(\overline{G})$ for some graph parameter $p(G)$. Viewing $\{G,\overline{G}\}$ as a decomposition of K_n allows us to generalize these theorems to decompositions of K_n with more than two factors. We determine the sum upper bound for independence number, domination number, edge independence number, maximum degree, edge chromatic number, and clique number. We also determine the extremal decompositions for the product lower bound for chromatic number.

One common way to study a graph parameter $p(G)$ is to examine the sum $p(G)+p(\overline{G})$ and product $p(G)\cdot p(\overline{G})$. A theorem providing sharp upper and lower bounds for this sum and product is known as a **Nordhaus-Gaddum theorem**. The original Nordhaus-Gaddum Theorem [12] dealt with chromatic number; there are now hundreds of analogous results for other parameters. Of the four possible bounds, the sum upper bound has attracted the most attention.

Many authors state when a bound in a Nordhaus-Gaddum theorem is an equality in terms of conditions on a graph. This often leads to awkward characterizations, as authors try to describe conditions on $\overline{G}$ as conditions on G. It is more convenient to think of G and $\overline{G}$ as a decomposition of K_n.

Definition 1. A **decomposition** of G is a set of nonempty subgraphs, called **factors**, whose edge sets partition $E(G)$. The subgraphs are said to **decompose** G. A **k-decomposition** of a graph G is a decomposition of G into k subgraphs. We use $\{G_1,...,G_k\}$ to denote a k-decomposition of G with factors G_i.

Aouchiche and Hansen [1] compiled a huge survey of Nordhaus-Gaddum theorems for 2-decompositions (see also [6] and [5]). Framing Nordhaus-Gaddum theorems in terms of decompositions naturally leads to the question of finding similar theorems for decompositions with more than two factors.

Definition 2. For a graph parameter p, let $p(k;G)$ denote the maximum of $\sum_{i=1}^{k} p(G_i)$ over all k-decompositions of G.

This idea was introduced by Plesnik for chromatic number [13]. It was studied by Furedi et al [10] for degeneracy, chromatic number, clique number, and list chromatic number. Their results on degeneracy were extended in [3], and other parameters were considered in [2].

Also in [3], it was observed that for the connectivity κ, edge-connectivity λ, minimum degree δ, and degeneracy $D(G)$, $\kappa(k, K_n) \leq \lambda(k, K_n) \leq \delta(k, K_n) \leq D(k; K_n)$, and these are all equalities for $1 \leq k \leq 4$. It was further conjectured that $\kappa(k, K_n) = \lambda(k, K_n) = \delta(k, K_n) = D(k; K_n)$ for all positive integers n and k.

Definitions of terms and notation not defined here appear in [4]. In particular, K_n is the complete graph of order n, and $K_{1,n-1}$ is the star of order n. Also, $\overline{G}$ is the complement of G. We use n for the number of vertices when the context is clear.

1 Independence, Matchings, and Domination

In this section, we determine $p(k; G)$ for parameters related to independence, matchings, and domination. We start with the independence number α.

Proposition 3. *For all positive integers n and k, $\alpha(k; K_n) = (k-1)n + 1$.*

Proof. Consider the decomposition $\{K_n, \overline{K}_n, ..., \overline{K}_n\}$. Then $\sum \alpha(G_i) = (k-1)n + 1$.
We use induction on order. Certainly $\alpha(k; K_1) = k$. Assume $\alpha(k; K_r) = (k-1)r + 1$, and let D be a decomposition of $G = K_{r+1}$. Consider the decomposition D' of $G - v$ formed by deleting v from each subgraph of D. If $\sum_{D'} \alpha(G_i) < (k-1)r + 1$, then $\sum_D \alpha(G_i) \leq (k-1)r + k = (k-1)(r+1) + 1$. If $\sum_{D'} \alpha(G_i) = (k-1)r + 1$, then by the pigeonhole principle, some vertex of K_r is contained in all k independent sets. Then v is contained in at most $k-1$ independent sets, so $\sum_D \alpha(G_i) \leq (k-1)r + 1 + (k-1) = (k-1)(r+1) + 1$. In either case, the result holds by induction. □

Note that $\{K_n, \overline{K}_n, ..., \overline{K}_n\}$ is not the only extremal decomposition. For $k = 2$, the extremal decompositions are the same as those for $\sum \omega(G_i) = n+1$ by complementation. More generally, if we have an extremal decomposition, we can produce another by either adding another $\overline{K}_n$, or adding a new vertex and joining it to all vertices of one factor.

Consider the domination number γ. Note that for any graph G, $\gamma(G) \leq \alpha(G)$, so $\gamma(G) + \gamma(\overline{G}) \leq \alpha(G) + \alpha(\overline{G}) \leq n + 1$. Jaeger and Payan [11] first proved that $\gamma(G) + \gamma(\overline{G}) \leq n + 1$. The extremal 2-decompositions are $\{K_n, \overline{K}_n\}$, as shown by Borowiecki [7] and Cockayne and Hedetniemi [8]. We can generalize these results to k factors.

Proposition 4. *For all positive integers n and k, $\gamma(k; K_n) = (k-1)n + 1$. The extremal k-decompositions are $\{K_n, \overline{K}_n, ..., \overline{K}_n\}$.*

Proof. We use induction on order. Certainly $\gamma(k; K_1) = k$. Assume $\gamma(k; K_r) = (k-1)r + 1$, and let D be a k-decomposition of $G = K_{r+1}$. Let v be a vertex not in all k minimum dominating sets of D (which must exist since not all factors are empty). Consider the decomposition D' of $G - v$ formed by deleting v from each subgraph of D. Now $\sum_D \gamma(G_i) \leq (k-1)r + 1 + k - 1 = (k-1)(r+1) + 1$.
Equality requires $D' = \{K_r, \overline{K}_r, ..., \overline{K}_r\}$. Now v only increases the domination number of a factor if it is isolated in that factor, so all edges incident with v must be in a single factor. That factor must be K_r, since adding all edges to another factor reduces its domination number to 1. Then $D = \{K_{r+1}, \overline{K}_{r+1}, ..., \overline{K}_{r+1}\}$. □

The formula for edge independence number β depends on the edge chromatic number χ'. Note that $\chi'(K_n) = n$ for n odd and $\chi'(K_n) = n - 1$ for n even.

Proposition 5. *For all positive integers n and k, $\beta(k; K_n) = \left\lfloor \frac{n}{2} \right\rfloor \min\{k, \chi'(K_n)\}$.*

Proof. A decomposition containing $\min\{k, \chi'(K_n)\}$ copies of $\left\lfloor \frac{n}{2} \right\rfloor K_2$ shows that $\beta(k; K_n) \geq \left\lfloor \frac{n}{2} \right\rfloor \min\{k, \chi'(K_n)\}$. Equality must hold since $\beta(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$ and there are at most $\chi'(K_n)$ such factors. □

Next we consider the maximum degree Δ.

Proposition 6. *For all positive integers n and k, $\triangle(k;K_n)=\binom{n}{2}-\binom{n-k}{2}$.*

Proof. Consider the decomposition with $G_i=K_{1,\max(n-i,0)}$ and any extra edges distributed arbitrarily. Then $\sum\triangle(G_i)=\sum_{i=\max(n-k,0)}^{n-1}i=\binom{n}{2}-\binom{n-k}{2}$.
We use induction on order. If $k\geq n$, then $\sum_D\triangle(G_i)\leq\sum_D m(G_i)=\binom{n}{2}$. If $k<n$, assume $\triangle(k;K_r)=\binom{r}{2}-\binom{r-k}{2}$, and let D be a decomposition of $G=K_{r+1}$. Let v be a vertex that does not uniquely have maximum degree in any of the k subgraphs. Consider the decomposition D' of $G-v$ formed by deleting v from each subgraph of D. Then adding v to the subgraphs of D' increases each maximum degree by at most one. Then $\sum_D\triangle(G_i)\leq\binom{r}{2}-\binom{r-k}{2}+k=\binom{r+1}{2}-\binom{r+1-k}{2}$. The result holds by induction. □

When $k=2$, the upper bound is $2n-3$, and this is attained exactly when one factor has both a dominating vertex and a leaf. We will see that $\chi'(k;K_n)$ is closely related to $\triangle(k;K_n)$.

Theorem 7. *For all positive integers n and k,*

$$\chi'(k;K_n)=\begin{cases}\Delta(k;K_n)+1 & n-k\geq 2\,even\\ \Delta(k;K_n) & else\end{cases}.$$

Proof. Consider the decomposition with $G_i=K_{1,\max(n-i,0)}$, $1\leq i<k$, and $G_k=K_{\max(n-k+1,1)}$. Then $\sum\chi'(G_i)=\sum_{i=\max(n-k,0)}^{n-1}i=\binom{n}{2}-\binom{n-k}{2}=\Delta(k;K_n)$ unless $\max(n-k+1,1)$ is odd and at least 3. Then $\chi'(G_k)=\triangle(G_k)+1$, so $\sum\chi'(G_i)=\Delta(k;K_n)+1$.
To show that this is an equality, we use induction on $n+k$. When $k=1$, $G_1=K_n$, and the result is obvious. When $1\leq n\leq 2$, the result is also obvious. Consider a k-decomposition $D=\{G_1,...,G_k\}$ of K_n with $k\geq 2$ and $n\geq 3$, and assume the result holds for smaller values of $n+k$.
Let v have degree $\Delta(G_1)$ in G_1. Delete v and all edges of G_1 from D, and redistribute any edges of G_1-v to other factors. This produces a $k-1$-decomposition $D'=\{H_2,...H_k\}$ of K_{n-1} so that $G_i\subseteq H_i$ for $2\leq i\leq k$. Note that $\chi'(G_i)\leq d_{G_i}(v)+\chi'(G_i-v)$ since each edge incident with v could be colored with a distinct color if necessary. Summing, we see $\sum_D\chi'(G_i)\leq n-1+\sum_{D-v}\chi'(G_i-v)\leq n-1+\sum_{D'}\chi'(H_i)$. Note that D' has $(n-1)-(k-1)=n-k$. If $n-k\geq 2$ is even, then by induction,

$$\sum_D\chi'(G_i)\leq n-1+\sum_{D'}\chi'(H_i)\leq n-1+\binom{n-1}{2}-\binom{n-k}{2}+1=\binom{n}{2}-\binom{n-k}{2}+1.$$

Otherwise,

$$\sum_D\chi'(G_i)\leq n-1+\sum_{D'}\chi'(H_i)\leq n-1+\binom{n-1}{2}-\binom{n-k}{2}=\binom{n}{2}-\binom{n-k}{2}.$$

□

When $k=2$ and n is even, the bound is $2n-2$, and it is attained by $\{K_{1,n-1},K_{n-1}\}$, and up to $\frac{n-3}{2}$ independent edges can be deleted from the K_{n-1}.

2 Clique Number and Chromatic Number

The following result on the clique number was proved by Furedi et al [10] using induction on $n+k$. We present a shorter proof of their result.

Theorem 8. (Furedi et al [10]) *For all positive integers n and k with $n \geq \binom{k}{2}$, $\omega(k; K_n) = n + \binom{k}{2}$.*

Proof. The decomposition of the line graph $L(K_k)$ into k copies of K_{k-1} has $n = \binom{k}{2}$ and $\sum \omega(G_i) = k(k-1) = n + \binom{k}{2}$. When $n > \binom{k}{2}$, we increase $\sum \omega(G_i)$ by one by iteratively adding a new vertex and joining it to all vertices of any given clique.

Consider a k-decomposition of K_n. The subgraphs of the factors that are critical with respect to ω are cliques; call them G_i. Now two cliques overlap on at most one vertex. Thus there are at most $\binom{k}{2}$ pairs of cliques that overlap. When r cliques overlap at vertex v, $\binom{r}{2}$ pairs of cliques overlap at v, and v is counted r times in $\sum \omega(G_i)$. Thus $\sum \omega(G_i)$ is maximized when each vertex is contained in one or two cliques, so $\sum \omega(G_i) \leq n + \binom{k}{2}$. $\square$

Note that $\sum \omega(G_i) = n + \binom{k}{2}$ requires that each pair of cliques overlap on distinct vertices, so the only extremal decompositions are those described in the proof.

It is immediate that $\chi(k; K_n) \geq \omega(k; K_n)$. In 1978, Jan Plesnik made the following conjecture.

Conjecture 9. (Plesnik's Conjecture [13]) *For $n \geq \binom{k}{2}$, $\chi(k; K_n) = n + \binom{k}{2}$.*

For $k = 2$, this is just the Nordhaus-Gaddum theorem. Plesnik proved the conjecture for $k = 3$ and determined a recursive upper bound of $\chi(k; K_n) \leq n + t(k)$, where $t(2) = 1$ and $t(k) = \sum_{i=2}^{k-1} \binom{k}{i} t(i)$. Thus $t(3) = 3$ and $t(4) = 18$. This implies a worse explicit bound of $\chi(k; K_n) \leq n + 2^{\binom{k+1}{2}}$. In 1985, Timothy Watkinson [14] improved this upper bound to $\chi(k; K_n) \leq n + \frac{k!}{2}$. In 2005, Furedi et al [10] proved an improved upper bound for large k of $\chi(k; K_n) \leq n + 7^k$. All of these bounds remain far from Plesnik's conjecture, however.

3 Product Lower Bound for Chromatic Number

We now consider the product lower bound for chromatic number, which is part of the original Nordhaus-Gaddum Theorem.

Proposition 10. (Nordhaus/Gaddum [12]) *For any graph G with order n, $\chi(G) \cdot \chi(\overline{G}) \geq n$.*

Proof. Let $\chi(G) = a$, and for a given a-coloring of G, let H be a complete a-partite graph that contains G. Then $\overline{H}$ is a disjoint union of complete graphs, one of which contains at least $\frac{n}{a}$ vertices. Then $\chi(G) \cdot \chi(\overline{G}) \geq \chi(H) \cdot \chi(\overline{H}) \geq a \cdot \frac{n}{a} = n$. $\square$

The extremal graphs were determined by Finck [9], with a proof whose length is about 1 page. His characterization was somewhat inelegant; a nice characterization for the extremal decompositions follows.

Corollary 11. [9] *A 2-decomposition $\{G, \overline{G}\}$ of K_n has $\chi(G) \cdot \chi(\overline{G}) = n$ if and only if $\{bK_a, aK_b\} \subseteq \{G, \overline{G}\}$, for $a, b \in \mathbb{N}$ with $ab = n$.*

Proof. Equality in the chain of inequalities in the previous proof requires that $b = \frac{n}{a}$ is a natural number, and every component of $\overline{H}$ is K_b. Thus $aK_b \subseteq \overline{G}$. Reversing the roles of the factors, we similarly see that $bK_a \subseteq G$. It is clear that both are always possible (e.g. by arranging the vertices in an $a \times b$ array). $\square$

Plesnik [13] determined the same bound holds for k factors. We also describe the extremal decompositions.

Theorem 12. *For any decomposition $\{G_1, ..., G_k\}$ of K_n, $\prod \chi(G_i) \geq n$.*
A k-decomposition $\{G_1, ..., G_k\}$ of K_n has $\prod \chi(G_i) = n$ if and only if there are a_i with $\prod a_i = n$ so that $\frac{n}{a_i} K_{a_i} \subseteq G_i$, for all i.

Proof. We use induction on k; the result is obvious when $k=1$. Assume that $\prod \chi(G_i) \geq n$ for any $r-1$-decomposition. Consider an r-decomposition $\{G_1, ..., G_r\}$ of K_n. Let $\chi(G_r) = a$. For a given a-coloring of G_r, let H_r be a complete a-partite graph that contains G_r, and $H_i = G_i - E(H_r)$. Then $\overline{H}_r$ is a disjoint union of complete graphs, one of which contains at least $\frac{n}{a}$ vertices. Then by induction,

$$\prod_{i=1}^{r} \chi(G_i) = \chi(G_r) \cdot \prod_{i=1}^{r-1} \chi(G_i) \geq \chi(H_r) \cdot \prod_{i=1}^{r-1} \chi(H_i) \geq a \cdot \frac{n}{a} = n.$$

Equality in the previous chain of inequalities requires that $\frac{n}{a}$ is a natural number, and every component of $\overline{H}_r$ is $K_{n/a}$. Applying this argument repeatedly, we eventually find that if $\chi(G_1) = a_1$, then $\frac{n}{a_1} K_{a_1} \subseteq G_1$. By ordering the factors differently, we can similarly see that if $\chi(G_i) = a_i$, $\frac{n}{a_i} K_{a_i} \subseteq G_i$, for all i. □

Note that $\frac{n}{a_i} K_{a_i} \subseteq G_i$, for all i is always possible. Arrange n vertices in a $a_1 \times ... \times a_k$ grid so that the sets of vertices that only vary in the i^{th} coordinate induce cliques in G_i. This is the only possible construction since any two cliques of distinct factors have at most one common vertex, hence exactly one.

References

[1] M. Aouchiche and P. Hansen, A survey of Nordhaus-Gaddum type relations, Discrete Appl. Math. 161 4-5 (2013), 466-546.

[2] A. Bickle, The k-Cores of a Graph, PhD thesis, Western Michigan University (2010).

[3] A. Bickle, Nordhaus-Gaddum theorems for k-decompositions, Congr. Numer. 211 (2012), 171-183.

[4] A. Bickle, Fundamentals of Graph Theory, AMS (2020).

[5] A. Bickle, Extremal Decompositions for Nordhaus-Gaddum Theorems, 2021+. Submitted.

[6] A. Bickle and A. White, Nordhaus-Gaddum results for genus, Discrete Math. 313 6 (2013), 824-829.

[7] M. Borowiecki, On the external stability number of a graph and its complement, Prace Naukowe Inst. Mat. Politechniki Wroclawskiej, 12 (1976), 39-43.

[8] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in graph, Networks 7 (1977), 247-261.

[9] H. J. Finck, On the chromatic numbers of a graph and its complement. Theory of Graphs (Proc. Colloq., Tihany, 1966) Academic Press, New York (1968), 99-113.

[10] Z. Furedi, A. Kostochka, M. Stiebitz, R. Skrekovski, and D. West, Nordhaus-Gaddum-type theorems for decompositions into many parts. J. Graph Theory 50 (2005), 273-292.

[11] F. Jaeger and C. Payan, Relations du type Nordhaus-Gaddum pour le nombre d'absorption d'un graphe simple, C. R. Acad. Sci. Paris Ser. A, 274 (1972), 728-730.

[12] E. A. Nordhaus and J. Gaddum, On complementary graphs, Amer. Math. Monthly 63 (1956), 175–177.

[13] J. Plesnik, Bounds on chromatic numbers of multiple factors of a complete graph. J. Graph Theory 2 (1978), 9-17.

[14] T. Watkinson, A theorem of the Nordhaus-Gaddum class. Ars. Combin. 20 (1985), 35-42.