2-Tone Coloring and Petersen Covers

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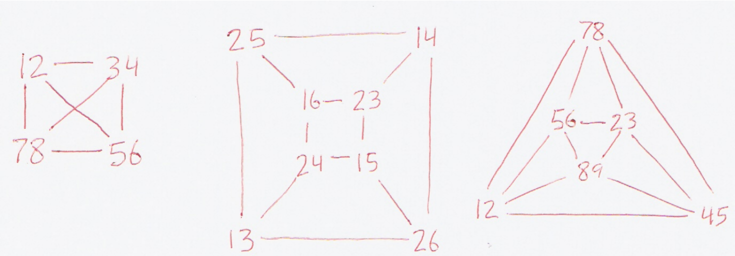
Definition

A 2-tone coloring of a graph assigns two colors to each vertex of a graph so that adjacent vertices have no common colors and vertices at distance two have at most one common color.

The label of a vertex is the pair of colors on a vertex.

A graph is 2-tone k-colorable if it can be 2-tone colored with k colors.

The 2-tone chromatic number $\tau_2(G)$ of a graph is the minimum number of colors in any 2-tone coloring.



Clearly $\tau_2(K_n) = 2n$.
Thus $2\omega(G) \le \tau_2(G) \le 2n$.
Also, $\tau_2(G) \ge \frac{2n}{\alpha(G)}$.

Theorem

We have $\tau_2(G) \le \chi(G) + \chi(G^2)$.

Proof.

Combining a proper vertex coloring of G and a proper coloring of G^2 yields a 2-tone coloring of G. $\square$

Complete Multipartite Graphs

Theorem

For the complete multipartite graph $K_{a_1,a_2,\ldots,a_r}$,

$$\tau_2\left(K_{a_1,\ldots,a_r}\right) = \sum_{i=1}^{r} \left\lceil \frac{1+\sqrt{1+8a_i}}{2} \right\rceil .$$

Proof.

Each partite set cannot use any color in common with any other. In a partite set, we need need r colors, where $\binom{r}{2} \geq a_i$. Solving for r gives the formula. $\square$

Stars

Corollary

For the nontrivial star $K_{1,s}$,

$$\tau_2(K_{1,s}) = \left\lceil \frac{5+\sqrt{1+8s}}{2} \right\rceil.$$

Proof.

$$\tau_2(K_{1,s}) = 2 + \left\lceil \frac{1+\sqrt{1+8s}}{2} \right\rceil = \left\lceil \frac{5+\sqrt{1+8s}}{2} \right\rceil.$$

□

Corollary

Let G have maximum degree $\triangle$. Then

$$\tau_2(G) \geq \left\lceil \frac{5+\sqrt{1+8\triangle}}{2} \right\rceil.$$

Bridges

Theorem

Let G be a graph with a bridge $e = uv$. Let F_1 and F_2 be the components of $G - e$ containing u and v, respectively, and let $H_1 = G[F_1 \cup v]$, and $H_2 = G[F_2 \cup u]$. Then $\tau_2(G) = \max\{\tau_2(H_1), \tau_2(H_2)\}$.

Proof.

The two subgraphs can be colored to agree on the bridge without conflict. $\square$

Theorem

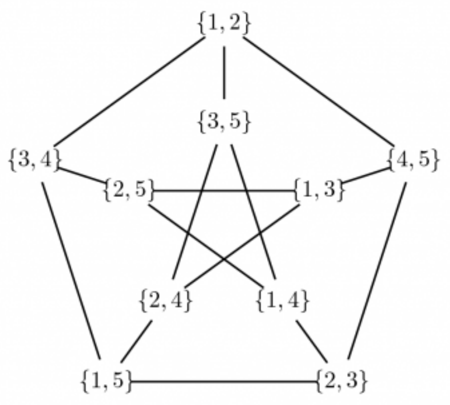
Let T be a nontrivial tree with maximum degree $\triangle$. Then

$$\tau_2(T) = \left\lceil \frac{5 + \sqrt{1 + 8\triangle}}{2} \right\rceil.$$

Proof.

By the theorem on bridges, the 2-tone chromatic number of a tree is the maximum of the 2-tone chromatic numbers of all the stars it contains. $\square$

Consider forming a graph whose vertices are the ten possible labels for a 2-tone 5-coloring and all possible edges are added. This graph is the Petersen graph. In fact, the Petersen graph can be defined with just this labeling. Thus its 2-tone chromatic number is five, so any subgraph of the Petersen graph is 2-tone 5-colorable.



Theorem

If a graph G has diameter at most four, then G is 2-tone 5-colorable $\Longleftrightarrow$ G is a subgraph of the Petersen graph.

Proof.

We have seen that every subgraph of the Petersen graph is 2-tone 5-colorable.
If G is 2-tone 5-colorable, it is easily seen that two vertices with the same label must be distance at least five apart, since otherwise $C_4 = K_{2,2}$ could be colored with five colors. Thus if G has diameter at most four, then no label can be repeated, so it is a subgraph of the Petersen graph. □

Definition

A 2-chord of a cycle to be a pair of vertices of the cycle whose distance is two.

Theorem

The Petersen Graph has no 7-cycle.

Proof.

Suppose it has a 7-cycle. Then C_7 has seven 2-chords, and each must have at most one common color. Now 14 colors are used with repetition, and each color can be used at most three times. Then four colors are used three times, and one is used twice. But this implies that C_7 has at least eight 2-chords. $\square$

Cycles

Theorem

For the cycle C_n,

$$\tau_2(C_n) = \begin{cases} 6 & n = 3,4,7 \\ 5 & \text{else} \end{cases}.$$

Proof.

Certainly $\tau_2(C_n) \geq 5$. Now the cycles of length 3, 4, and 7 are not subgraphs of the Petersen graph. Now $C_3 = K_3$, and C_4 can have nonadjacent vertices share a color, so $\tau_2(C_3) = \tau_2(C_4) = 6$. Now C_7 can be labeled as below. The cycles of length 5, 6, 8, and 9 are subgraphs of the Petersen graph, with labellings below represented as broken cycles. □

Cycles

Proof.

$$-12-34-51-23-45-$$

$$-12-34-15-32-14-35-$$

$$-12-34-56-13-24-35-46-$$

$$-12-34-15-23-14-25-13-45-$$

$$-12-34-15-32-14-25-13-24-35-$$

Finally, for $n \geq 10$, the cycle can be constructed by breaking and attaching together cycles of length 5, 6, 8, and 9, which can be done because the labellings above agree on the first three vertices. $\square$

Regular Graphs

Theorem

If G is r-regular, $r \geq 2$, then $\tau_2(G) \leq r^2 + r$.

Proof.

Let G be r-regular, $r \geq 2$. Note that each vertex of G has at most $r + r(r-1) = r^2$ other vertices within distance two, so $\triangle(G^2) \leq r^2$. Hence

$$\tau_2(G) \leq \chi(G) + \chi(G^2) \leq (1 + \triangle(G)) + (1 + \triangle(G^2)) \leq (1+r) + (1+r^2).$$

By Brooks' Theorem, the middle inequality can be an equality only when G or G^2 are complete or an odd cycle. Now for $r \geq 2$, $\tau_2(K_r) = 2r \leq r^2 + r$ and $\tau_2(C_n) \leq 6 \leq r^2 + r$. Now G^2 cannot be a noncomplete odd cycle.

□

Proof.

The only case in which we might have $\chi(G^2) = (1+r^2)$ is when G^2 is complete with order $1+r^2$. But this means that G must have diameter 2 and girth 5, so G is a Moore graph. But Hoffman and Singleton [1960] showed that the only Moore graphs with girth 5 occur when $r = 2, 3, 7$, and possibly 57 (this case is undecided). The case $r = 3$ is the Petersen graph (PG), which has $\tau_2(PG) = 5 < 12$. The case $r = 7$ is the Hoffman-Singleton graph (HS), for which $\chi(HS) = 4$, so $\tau_2(HS) \leq 4 + 49 < 56$. Finally, Borodin and Kostochka [1977] showed that for a K_4-free graph, $\chi(G) \leq \left\lceil \frac{3}{4}(\triangle(G)+1) \right\rceil$, so if a graph M satisfies the final case $r = 57$, then $\chi(M) \leq 44$, so $\tau_2(M) \leq 44 + 57^2 < 57 + 57^2$. □

Conjecture

Let G have maximum degree $\triangle$. Then $\tau_2(G) \leq 2\triangle + 2$, with equality only if G contains $K_{\triangle+1}$ or for $\triangle = 2$, C_4 or C_7.

Theorem

For the join $G + H$,

$$\tau_2(G+H) \geq \tau_2(G) + \tau_2(H).$$

If G and H have diameter at most 2, then this is an equality.

Proof.

No common color can be used in both subgraphs G and H of the join. If G and H both have diameter at most 2, then so does $G + H$. Therefore combining minimal colorings for G and H creates no conflict, so the bound is achieved. $\square$

The bound may not be exact because vertices that have distance greater than two in G will have distance two in $G + H$.

Theorem

Let $n_1 = n(G)$, $n_2 = n(H)$. *Then*

$$\tau_2(G+H) \geq \tau_2(K_{n_1,n_2}) = \sum_{i=1,2} \left\lceil \frac{1+\sqrt{1+8n_i}}{2} \right\rceil .$$

Proof.

$G+H$ contains K_{n_1,n_2} as a subgraph. □

This bound appears to be good for sparse graphs, but it is unclear exactly when it is an equality.

Wheels

Theorem

$$\tau_2(W_n) = \begin{cases} 7 & n = 5,6,8,9 \\ 8 & n = 3,4,7,10-15 \\ r+2 & \binom{r-1}{2} < n \leq \binom{r}{2}, r \geq 6 \end{cases}.$$

Proof.

(sketch) The center vertex requires two colors, which cannot be used on any other vertex. The first lower bound yields $\tau_2(W_n) = 2 + \tau_2(C_n)$ for $3 \leq n \leq 9$ since these cycles can be colored without repeating a label. The second lower bound requires that every vertex of the cycle have a distinct label, so it requires at least r colors if n satisfies $\binom{r-1}{2} < n \leq \binom{r}{2}$.

□

Proof.

Consider explicit colorings of cycles where no pair is repeated. First consider the following broken cycle

$$-12-(56)-34-25-(36)-14-23-45-(26)-13-(46-15)-24-16-35-$$

Without the pairs in parentheses, we have a ten-cycle. The pairs in parentheses can be 'inserted' into the cycle, preserving the necessary properties. (The two pairs in a single set of parentheses must be inserted at the same time.) This provides constructions up to the 15-cycle, for which all pairs formed from six colors are used. We next use induction on r to prove the existence of constructions for larger values of n. Assume that for $r \geq 6$, there exists a 2-tone coloring of the cycle with $\binom{r}{2}$ vertices using r colors, so that each possible pair is used exactly once. We want to insert new pairs of colors in between some of the existing pairs. Allowing color $r+1$ adds r new pairs to insert. $\square$

Proof.

We model this situation with a bipartite graph as follows. One partite set is the r new labels to be added. The other partite set is the $\binom{r}{2}$ possible locations for insertion. An edge joins two vertices if the particular label can be inserted in the particular location. We seek a maximum matching in this bipartite graph.

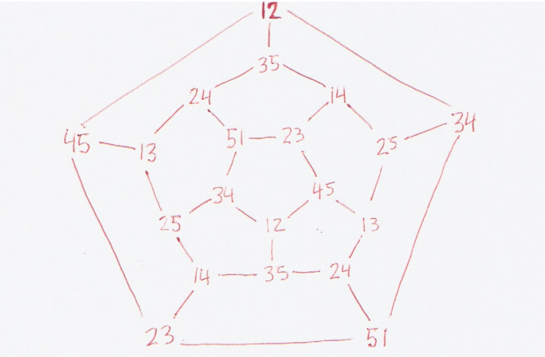
It is straightforward to check that vertices in the location partite set have degree $r-4$, and each vertex of the label partite set has degree $\binom{r}{2} - 2(r-1) = \frac{1}{2}(r-1)(r-4)$.

This implies that the bipartite graph satisfies Hall's condition, so it has a maximum matching. Thus the new pairs can be successively inserted up to a cycle of length $\binom{r+1}{2}$. By induction, we have constructions for all $n \geq 15$.

Our constructions achieve one of the lower bounds in all but the case $n = 10$, for which our construction is one larger. They cannot be achieved in this case since the Petersen graph is non-Hamiltonian.

Definition

A Petersen cover is a covering graph of the Petersen graph. That is, it is a graph for which there is an onto homomorphism f from G to the Petersen graph with the property that for each vertex v of G the neighborhood of v maps bijectively onto the neighborhood of $f(v)$.



Theorem

A cubic graph G has $\tau_2(G) = 5$ if and only if G is a Petersen cover.

Proof.

Let G be a cubic graph G with $\tau_2(G) = 5$. Then given a 2-tone 5-coloring of G, map all the vertices with the same label to the vertex of the Petersen graph with that label. Now no edge of G is mapped to a nonedge of the Petersen graph, since then it would violate the labeling. No adjacent edges are mapped to the same edge, since then there would be two vertices at distance two with the same label. Thus G is a Petersen cover.
Let G be a Petersen cover. Label all the vertices that map to a given vertex of the Petersen graph under the homomorphism with the same label. This produces a 2-tone coloring of G. $\square$

Corollary

A Petersen cover has order $n = 10k$, k a positive integer.

Proof.

Let G be a Petersen cover. Suppose k vertices receive label X in a 2-tone 5-coloring, and one of them is v. Then every other label appears exactly once amongst the vertices within distance two of v, as in the Petersen graph. Further, none of these vertices is within distance two of any other vertex with label X, since such vertices must be distance at least five apart. By considering another label, we see that the number of vertices receiving each label is the same. Thus the order of G is a multiple of ten. □

Corollary

If G is cubic, with order $n \neq 10k$, it is not 2-tone 5-colorable.

Definition

Let G be a graph containing vertices u, v, w, x, edges uv, wx, and not containing edges uw, vx. A 2-switch is the operation that deletes edges uv and wx and adds edges uw and vx.

Theorem

G is a Petersen cover if and only if it can be obtained by starting with k disjoint copies of the Petersen graph and performing some number of 2-switches on pairs of edges that join vertices with the same labels.

Proof.

Consider a graph produced by this process. The edges switched by a 2-switch must still map to the same edge of the Petersen graph, so the graph is a Petersen cover.

Let G be a Petersen cover. Consider aligning the vertices that map to the same vertex of the Petersen graph in a column of k levels, with one vertex of each type per level. By performing 2-switches, it is possible to produce edges that all join vertices of the same level, separating out k copies of the Petersen graph. Reversing the sequence of 2-switches, we see that G can be constructed in this way. $\square$

Theorem

Let G be a Petersen cover. Then
1. G does not contain a bridge.
2. If G contains a minimal 2-edge-cut, then performing a 2-switch on those edges separates G into two components which are two smaller Petersen covers.
3. If G has a minimal 3-edge-cut, it is trivial.

Proof.

(1 only) Performing a 2-switch on a pair of edges, of which at least one is a bridge produces another pair of edges, at least one of which is a bridge. Thus no bridge can be produced by the process described in the previous theorem. □

Critical Graphs

Definition

A graph G is t-tone k-critical if $\tau_t(G) = k$ and for any proper subgraph H of G, $\tau_t(H) < k$.

Theorem

Let G be a graph containing P_3 with vertices u and v not adjacent and let $e = uv$. Then $\tau_2(G) - \tau_2(G - e) \leq 1$.

The proof breaks into three cases depending on the labels of u and v.

Theorem

Aside from $K_{1,4}$ and the 3-, 4-, and 7-cycles, any 2-tone 6-critical graph G has $\triangle(G) = 3$, $\delta(G) = 2$, and is 2-connected.

We can characterize the 6-critical graphs in one particular family of graphs. The theta graph $\theta_{i,j,k}$ is formed by taking paths of lengths i, j, k and identifying them at their end-vertices. It necessarily contains three cycles of lengths $a = i + j$, $b = i + k$, $c = j + k$. We will use (i, j, k) for $\theta_{i,j,k}$.

Theorem

The theta graph $\theta_{1,2,2}$ has $\tau_2(\theta_{1,2,2}) = 7$, and $(3,3,3)$, $(3,3,5)$, $(3,3,6)$, $(4,4,4)$, $(4,4,5)$, and $(3,3,9)$, all have 2-tone chromatic number 6. For all other theta graphs, $\tau_2(\theta_{i,j,k}) = \max\{\tau_2(C_a), \tau_2(C_b), \tau_2(C_c)\}$, where $a = i + j$, $b = i + k$, $c = j + k$.

Theorem

There are infinitely many 2-tone 6-critical graphs.

Proof.

We know that such graphs exist, and all but $K_{1,4}$ contain cycles. Assume to the contrary that the number of 2-tone 6-critical graphs is finite. Let g be the maximum girth of all such graphs. It is well-known that there is a cubic graph with girth larger than g. If its order is not a multiple of ten, it is not a Petersen cover. If it is, subdivide an edge and join two copies of it by adding an edge between the two vertices of degree two. The resulting graph is not a Petersen cover. Either way, there is a cubic graph G of girth more than g that is not a Petersen cover. Then its 2-tone number is at least six, so it contains a 6-critical subgraph H, which also has girth more than g. This is a contradiction. $\square$

Conjecture

Let G be a cubic graph. Then
1. $\tau_2(G) = 8$ if and only if G contains K_4.
2. $\tau_2(G) = 7$ if and only if G contains $K_4 - e$ and not K_4.
3. $\tau_2(G) = 6$ if and only if G is not a Petersen cover and does not contain $K_4 - e$.

Thank you!

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