2-Tone Coloring of Joins and Products of Graphs

Allan Bickle

Western Michigan University

April 10, 2013

History of t-Tone Coloring

- Math 645 (Summer 2009) taught by Ping Zhang - Class project of Nicole Fonger, Josh Goss, Ben Phillips, and Chris Segroves (FGPS) - assisted by Gary Chartrand
- Bickle and Phillips (BP) work on the topic Summer 2009-2011 - submit paper *"t-Tone Colorings of Graphs"* (2011)
- Bickle presents at conferences - 41st Southeastern (2010) and MIGHTY LI (2011)
- Cranston, Kim, and Kinnersly (CKK) submit the paper *"New Results in t-Tone Coloring* of Graphs" following up on our work (2011)
- Bal, Bennett, Dudek, and Frieze submit the paper "The t-tone chromatic number of random graphs" (2012)
- Thanks to Ben Phillips for introducing me to the topic.
- Thanks to Drs. Gary Chartrand, Allen Schwenk, Doug West, and Ping Zhang for their advice.

Definitions

Definition

A 2-tone coloring of a graph assigns two colors to each vertex of a graph so that adjacent vertices have no common colors and vertices at distance two have at most one common color.
The label of a vertex is the pair of colors on a vertex.
A graph is 2-tone k-colorable if it can be 2-tone colored with k colors.
The 2-tone chromatic number $\tau_2(G)$ of a graph is the minimum number of colors in any 2-tone coloring.

Upper Bounds

Proposition

[FGPS 2009] We have $\tau_2(G) \le \chi(G) + \chi(G^2)$.

Theorem

[BP 2010] Let a graph G have maximum degree $\Delta = \Delta(G) \ge 2$. *Then* $\tau_2(G) \le \Delta^2 + \Delta$.

Theorem

[CKK 2011] We have $\tau_2(G) \le \lceil 2 + \sqrt{2} \rceil \Delta$.

Theorem

[Bickle 2011] Let a graph G have maximum degree $\Delta = \Delta(G) \ge 2$. *Then* $\tau_2(G) \le 2\Delta - 1 + \left\lceil \frac{1 + \sqrt{1 + 8\Delta(\Delta - 1)}}{2} \right\rceil$.

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Joins

For the 2-tone chromatic number of a join of graphs, we have the following partial results.

Theorem

For the join $G+H$, $\tau_2(G+H) \geq \tau_2(G) + \tau_2(H)$.
If G and H have diameter at most 2, then this is an equality.

Proof.

The inequality follows since no common color can be used in both subgraphs G and H of the join. If G and H both have diameter at most 2, then so does $G+H$. Therefore combining minimal colorings for G and H creates no conflict, so the bound is achieved. □

The bound may not be exact because vertices that have distance greater than two in G will have distance two in $G+H$. The converse is false, as for example W_6 achieves the bound even though the 6-cycle has diameter 3.

Pair Coloring

Thus for any factor in a join, the vertices must have a 2-tone coloring with the additional restriction that each label must be distinct. This motivates the following definition.

Definition

A pair k-coloring is a 2-tone k-coloring in which every label is distinct. A graph is pair k-colorable if it has a pair k-coloring. The pair chromatic number of a graph G, $pc(G)$, is the smallest k for which it has a pair k-coloring.

Some results on the pair chromatic number are immediate. We have $pc(G) \geq \tau_2(G)$, and if $diam(G) \leq 2$, then this is an equality. Hence it is an equality for almost all graphs. If H is a subgraph of G, then $pc(H) \leq pc(G)$. It is not difficult to show that $pc(G+e) - pc(G) \leq 1$. It is also straightforward to see that $pc(G+H) = pc(G) + pc(H)$.

Pair Coloring

A graph G is pair k-colorable $\iff$ it is contained in $L_k = L(K_k)$. Thus if $n > \binom{k}{2}$, $pc(G) > k$. Equivalently, $pc(G) \geq \frac{1+\sqrt{1+8n}}{2}$. Thus $pc\left(\overline{K}_n\right) = \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$. This also implies that given $n_1 = n(G)$ and $n_2 = n(H)$,

$$\tau_2(G+H) \geq \tau_2(K_{n_1,n_2}) = \sum_{i=1,2} \left\lceil \frac{1+\sqrt{1+8n_i}}{2} \right\rceil .$$

since $G+H$ contains K_{n_1,n_2} as a subgraph. This bound appears to be good for sparse graphs, but it is unclear exactly when it is an equality.

Pair Coloring

Theorem

Let G have degeneracy $k \leq n-1$. Then $\left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil \leq pc(G) \leq 2k + \left\lceil \frac{1+\sqrt{1+8(n-k)}}{2} \right\rceil$.

Proof.

The lower bound has already been justified. Color G with a construction sequence. Each vertex v has $j \leq k$ neighbors which exclude at most $2j$ colors. There are at most $n-j-1$ labels that have already been used on non-neighbors of v. Thus we need r extra colors, where $\binom{r}{2} \geq n-j$. Solving, we find $r \geq \frac{1+\sqrt{1+8(n-j)}}{2}$. Thus we need at most $2j + \left\lceil \frac{1+\sqrt{1+8(n-j)}}{2} \right\rceil$ colors to label v, which is maximized when $j = k$. □

The upper bound is attained for the graph $K_k + \overline{K}_{n-k}$. Since forests are exactly the 1-degenerate graphs, we have the following corollary.

Corollary

Let F be a forest. Then

$$\left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil \leq pc(F) \leq 2 + \left\lceil \frac{1+\sqrt{1+8(n-1)}}{2} \right\rceil.$$

Thus there are usually three possible values for the pair chromatic number of a forest, but there are only two for $n = \binom{r}{2} + 1$, $r \geq 2$. Note that stars attain the upper bound. Characterizing the trees that attain the upper bound may be possible, but distinguishing between the other two values appears difficult.

Pair Coloring

Proposition

Let F be a forest and let r be the smallest integer such that $n \leq \binom{r}{2} + 1$. If $\Delta(F) = \binom{r-1}{2}$, then $pc(F) \leq 1 + \left\lceil \frac{1+\sqrt{1+8(n-1)}}{2} \right\rceil$.

Proof.

The result is easily checked for $1 \leq n \leq 7$. Let F be a forest with order $n \geq 8$ and let $r \geq 5$ be the smallest integer such that $n \leq \binom{r}{2} + 1$. Add edges if necessary to form a tree T with the same maximum degree. Let v be a vertex with degree $\binom{r-1}{2}$, which WLOG receives label 12. Then its $\binom{r-1}{2}$ neighbors must receive all possible labels from $\{3, 4, \ldots, r+1\}$. Now F has at most $r-1$ vertices remaining, and $2(r-1)$ labels left. Label the remaining vertices with a construction sequence. □

Proof.

If the i^{th} vertex labeled is adjacent to a neighbor of v, then there are at least
$2(r-1)-(i-1)-4=2r-i-5\geq 2r-(r-1)-5=r-4>0$
labels remaining. If the i^{th} vertex labeled is not adjacent to a neighbor of v, then its neighbor u must have the color 1 or 2, and either excludes $r-1$ possible labels. The other color on u excludes one more label. Since u uses one of the labels already excluded, the preceding $i-1$ vertices exclude at most $i-2$ labels. Thus there are at least $2(r-1)-(r-1)-1-(i-2)=r-i\geq r-(r-1)=1$ labels remaining. Thus $r+1$ colors suffice to label T. □

Pair Coloring

Corollary

Let G have degeneracy $k \leq n-1$. Then

$$\left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil \leq pc(G) \leq \max_{0 \leq j \leq k} \left\{ 2j + \left\lceil \frac{1+\sqrt{1+8(n_j - j)}}{2} \right\rceil \right\}.$$

Proof.

Color G with a construction sequence. Let the k-core of G have order n_k. If vertex v is the i^{th} vertex colored and $j = C(v)$ is the core number of v, then neighbors of v exclude at most $2j$ colors. There are at most $n_j - j - 1$ labels that have already been used on non-neighbors of v. Thus we need r extra colors, where $\binom{r}{2} \geq n_j - j$. Solving, we find $r \geq \frac{1+\sqrt{1+8(n_j-j)}}{2}$. Thus we need at most $2j + \left\lceil \frac{1+\sqrt{1+8(n_j-j)}}{2} \right\rceil$ colors to label v. Maximize over j. □

Pair Coloring

Note that this is no improvement for monocore graphs, but may be an improvement otherwise. For regular graphs, the following corollary is an improvement.

Corollary

Let G be a connected graph with maximum degree $\Delta \geq 1$. Then
$$\left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil \leq pc(G) \leq 2\Delta - 1 + \left\lceil \frac{1+\sqrt{1+8(n-\Delta+1)}}{2} \right\rceil .$$

Proof.

Since G is connected, $G-e$ is $\Delta - 1$-degenerate. By an earlier theorem, $pc(G-e) \leq 2(\Delta-1) + \left\lceil \frac{1+\sqrt{1+8(n-(\Delta-1))}}{2} \right\rceil$. Lastly, adding e back requires at most one more color. □

Definition

A graph G is pair k-critical if for any proper subgraph H of G, $pc(H) < pc(G) = k$.

For small values of k, it is possible to list all such graphs.

k	Pair k-Critical Graphs
2	K_1
3	$\overline{K}_2$
4	$\overline{K}_4$, K_2
5	$\overline{K}_7$, P_3

It is considerably more difficult to determine all pair 6-critical graphs. We consider this problem below. Note that for any k, there is a finite number of pair k-critical graphs.

Proposition

Any graph G has finitely many critical forbidden subgraphs.

Proof.

Let G have order n. Then $\overline{K}_{n+1}$ is a critical forbidden subgraph of G, so any other critical forbidden subgraph must have order at most n. There are finitely many such graphs, some subset of which are not subgraphs of G. Some subset of these are critical. □

There is another reason why pair colorings are interesting. Consider labeling the vertices of a complete graph K_n with 1 to n. Then each edge can be labeled with the pair of labels of its vertices. Each possible label occurs exactly once. Thus a pair k-coloring of a graph corresponds to a (usually different) edge-induced subgraph of K_k. Thus we can transform a question on pair k-coloring of a disconnected graph into a question on decomposition (or packing) of a complete graph. As decompositions have been widely studied, results on them can be applied to pair coloring.
Some examples of graphs and corresponding subgraphs for their minimal colorings are given in the following table.

Graph	Corresponding Subgraph
K_n	nK_2
C_4	$2P_3$
C_5	C_5
C_6	$K_{2,3}$
C_8	W_4
C_9	$K_5 - e$
P_4	P_5
L_n	K_n
$K_{\binom{r}{2},\binom{s}{2}}$	$K_r \cup K_s$

For example, consider a union of complete graphs. Each clique must have no common colors on its vertices. Thus its coloring corresponds to a matching in a complete graph. We employ the following lemma.

Pair Critical Graphs

Lemma

(adapted from de Werra 1971 and McDiarmid 1972)
Let $a_1, \ldots, a_k$ *be integers with* $1 \leq a_1 \leq \ldots \leq a_k \leq \lfloor \frac{n}{2} \rfloor$. *If* $\sum a_i \leq \binom{n}{2}$, *then* K_n *has a packing with matchings of sizes* a_i.

Proof.

The extreme case occurs when there are n or $n-1$ matchings of size $\lfloor \frac{n}{2} \rfloor$. Such a decomposition is well-known. Hence we suppose that we have a packing of K_n with matchings of sizes $1 \leq b_1 \leq \ldots \leq b_k \leq \lfloor \frac{n}{2} \rfloor$, where $\sum a_i = \sum b_i$. If these numbers are not all the same, there must be integers $i < j$ such that $b_i < a_i \leq a_j < b_j$. Form a subgraph H by merging matchings i and j together. H must have each component be a path or even cycle. Since the two sizes of the matchings were unequal, H must have a component path of odd length. Swapping the edges on this path in the matchings moves us closer to the goal. Applying this process repeatedly must achieve it. □

Pair Critical Graphs

Theorem

Let $a_1, \ldots, a_k$ be integers with $1 \leq a_1 \leq \ldots \leq a_k$ and $n = \sum a_i$. Then $pc\left(\bigcup_i K_{a_i}\right) = \max\left\{2a_k, \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil\right\}$.

Proof.

Both lower bounds are immediate. Let $N = \max\left\{2a_k, \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil\right\}$. By the lemma, K_N can be packed with matchings of sizes $a_1, \ldots, a_k$. Hence a pair N-coloring of $\bigcup_i K_{a_i}$ exists. □

Corollary

Let G be a disconnected graph with components G_i with orders $a_1, \ldots, a_k$, $1 \leq a_1 \leq \ldots \leq a_k$. Then $\max\left\{pc(G_i), \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil\right\} \leq pc(G) \leq \max\left\{2a_k, \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil\right\}$.

Hence graphs for which the lower inequality is strict are of interest. Such a graph must have pair chromatic number at least 6. All such critical graphs are listed in the table below. This follows since (1) none of these graphs are subgraphs of the Petersen graph and (2) the the corresponding decompositions do not pack K_5.

Graph	Corresponding Decomposition
$T_{3-3} \cup P_3$	$\{K_1 + 2K_2, P_3 \cup K_2\}$
$C_6 \cup 2K_2$	$\{K_{2,3}, 2K_2, 2K_2\}$
$C_5 \cup K_{1,3}$	$\{C_5, K_3 \cup K_2\}$
$2K_{1,3}$	$\{K_3 \cup K_2, K_3 \cup K_2\}$

Now we can determine all pair 6-critical graphs. We define some notation. Let $T_{a-b-\ldots-c}$ be a tree (caterpillar) with a spine having vertices of degree a, b, ..., c. Similarly, $T_{P_a-P_b-\ldots-P_c}$ is formed by appending paths of lengths a, b, ..., c to a path.

Theorem

There are exactly 11 pair 6-critical graphs, namely $\overline{K}_{11}$, $K_{1,4}$, $2K_{1,3}$, $T_{3-3} \cup P_3$, T_{3-P_4-3}, K_3, C_4, C_7, C_{10}, $C_6 \cup 2K_2$, $C_5 \cup K_{1,3}$.

Description	Pair 6-Critical Graphs
forests	$\overline{K}_{11}$, $K_{1,4}$, $2K_{1,3}$, $T_{3-3} \cup P_3$, T_{3-P_4-3}
cycles	K_3, C_4, C_7, C_{10}
disconnected	$C_6 \cup 2K_2$, $C_5 \cup K_{1,3}$

Proof.

[sketch] Certainly $\overline{K}_{11}$ is pair 6-critical, so any other such graph has order at most 10. Now $K_{1,4}$ is pair 6-critical, so any other such graph has maximum degree at most 3.
Consider forests. We have $2K_{1,3}$ 6-critical since any two vertices of the Petersen graph have distance at most two apart. Since the Petersen graph has a Hamiltonian path, no path is 6-critical.
Checking cases shows that no spider (exacy one vertex of degree three) is 6-critical. Checking cases when there are exactly two vertices of degree three at distance two apart does not find any 6-critical graph. When there are two adjacent vertices of degree 3, we find $T_{3-3} \cup P_3$ is 6-critical. Otherwise, checking cases shows that T_{3-P_4-3} is the only other 6-critical forest. □

Proof.

Certainly the only 6-critical cycles are K_3, C_4, C_7, C_{10}. Now consider graphs containing a single cycle. The cycle C_9 is not the basis for any 6-critical unicyclic graph. Checking cases for C_8 does not produce any new 6-critical graphs. For C_6, we first find the disconnected graph $C_6 \cup 2K_2$. Checking cases for connected 6-critical unicyclic graphs does not find any more. Starting with C_5, we see $C_5 \cup K_{1,3}$ is a disconnected 6-critical graph. Checking cases when we consider appening one, two, three, four, or five trees does not produce any new 6-critical graphs.

There are two 2-tone 6-critical theta graphs ($\theta_{3,3,3}$ and $\theta_{3,3,5}$) with at most order 10, but both contain $2K_{1,3}$. Checking cases shows that no graph formed by appending trees to a theta graph is 6-critical. □

Proof.

Consider adding another path to a theta graph. There are two possibilities. One produces two disjoint cycles, the other produces a subdivided K_4. Note first that any 6-critical graph can be produced by adding an edge between nonadjacent vertices of the Petersen graph and then deleting some number of edges. It is easily seen that this number must be at least four since adding the edge creates a 3-cycle, two 4-cycles, several 7-cycles, and two vertices of degree 4 that must be disrupted. Now two disjoint cycles must be 5-cycles, but checking cases does not produce any new 6-critical graphs. Checking cases organized by length of the longest cycle in a subdivided K_4 also does not produce any new 6-critical graphs.

Finally consider disconnected graphs that are not forests and not unicyclic. Begin by deleting a subgraph of small order and then considering what other component could be added to make the graph not pair 5-chromatic, and whether the resulting graph is 6-critical. Deleting one, two, or three vertices produces nothing new. Deleting four or more vertices eliminates all but at most one cycle. This exhausts the search. □

Pair Coloring of Cycles

A nontrivial result on pair coloring concerns cycles.

Theorem

We have

$$pc(C_n) = \begin{cases} 5 & n = 5, 6, 8, 9 \\ 6 & n = 3, 4, 7, 10 - 15 \\ \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil & n \geq 11 \end{cases}.$$

Proof.

There are two relevant lower bounds to consider. First, $pc(C_n) \geq \tau_2(C_n)$. This is exact for $3 \leq n \leq 9$ since the unique minimal colorings for all but C_7 do not repeat a pair and C_7 has a minimal coloring that does not repeat a pair.

The second lower bound requires that every vertex of the cycle have a distinct pair, so $pc(C_n) \geq \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$. □

Proof.

We now consider explicit colorings of cycles where no pair is repeated. First consider the following broken cycle.

$$-12-(56)-34-25-(36)-14-23-45-(26)-13-(46-15)-24-16-35-$$

Without the pairs in parentheses, we have a ten-cycle. The pairs in parentheses can be 'inserted' into the cycle, preserving the necessary properties. That is, we can subdivide some edges and assign previously unused pairs to the new vertices. The two pairs in a single set of parentheses must be inserted at the same time. This provides constructions up to the 15-cycle, for which all pairs formed from six colors are used.

We next use induction on r to prove the existence of constructions for larger values of n. Assume that for $r \geq 6$, there exists a 2-tone coloring of the cycle with $\binom{r}{2}$ vertices using r colors, so that each possible pair is used exactly once. We want to insert new pairs of colors in between some of the existing pairs. Allowing color $r+1$ adds r new pairs to insert. □

Proof.

We model this situation with a bipartite graph as follows. One partite set is the r new pairs to be added. The other partite set is the $\binom{r}{2}$ possible locations for insertion. An edge joins two vertices if the particular pair can be inserted in the particular location. We seek a maximum matching in this bipartite graph.

Since each pair is distinct, and each pair uses the new color $r+1$, a pair can be inserted as long as the other color is not one of the four used on the vertices between which the new vertex will be inserted. Thus each vertex in the location partite set has degree $r-4$. Each existing color is used $r-1$ times times on the cycle. Since the vertices on which it is used form an independent set, each excludes two locations, leaving $\binom{r}{2}-2(r-1)=\frac{1}{2}(r-1)(r-4)$ valid locations, so this is the degree of the vertices in this partite set. □

Pair Coloring of Cycles

Proof.

Consider a subset S of new pairs of order s, and let its neighborhood $N(S)$ have order n. Then $s\left[\frac{1}{2}(r-1)(r-4)\right] \leq n(r-4)$, so $n \geq s$. Thus the bipartite graph satisfies Hall's condition, so it has a maximum matching. Thus the new pairs can be successively inserted up to a cycle of length $\binom{r+1}{2}$. By induction, we have constructions for all $n \geq 15$.

Our constructions achieve one of the lower bounds in all but the case $n = 10$, for which our construction is one larger. The bound cannot be achieved in this case since the Petersen graph is non-Hamiltonian. □

Corollary

$$\tau_2(W_n) = \begin{cases} 7 & n = 5, 6, 8, 9 \\ 8 & n = 3, 4, 7, 10-15 \\ \left\lceil \frac{5+\sqrt{1+8n}}{2} \right\rceil & n \geq 11 \end{cases}.$$

Pair Coloring of Cycles

Another corollary follows immediately.

Corollary

For $n \geq 3$,

$$pc(P_n) = \begin{cases} 5 & 3 \leq n \leq 10 \\ \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil & n \geq 11 \end{cases}.$$

Proof.

The path requires at least five colors, and enough so that every vertex has a distinct pair. For $n \geq 11$, breaking the cycle constructed in the proof of the theorem for wheels yields the appropriate construction. For smaller values, the appropriate construction exists because the Petersen graph has a hamiltonian path. □

This implies:

Corollary

Consider the fan $F_n = P_n + K_1$, $n \geq 3$. Then

$$\tau_2(F_n) = \begin{cases} 7 & 3 \leq n \leq 10 \\ \left\lceil \frac{5+\sqrt{1+8n}}{2} \right\rceil & n \geq 11 \end{cases}.$$

$$\tau_2(P_n + K_2) = \begin{cases} 9 & 3 \leq n \leq 10 \\ \left\lceil \frac{9+\sqrt{1+8n}}{2} \right\rceil & n \geq 11 \end{cases}.$$

Conjecture

Let G be a 2-regular graph with $n \geq 7$. Then $\left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil \leq pc(G) \leq 1 + \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil$.

We can also consider 2-tone chromatic numbers for cartesian products of graphs. These are also difficult to determine in general. We first consider upper bounds.

Note that if G_i has maximum degree Δ_i and maximum core number/degeneracy D_i, then $\Delta(G_1 \times G_2) = \Delta_1 + \Delta_2$ and $D(G_1 \times G_2) = D_1 + D_2$ [Bickle 2010]. We apply the technique of [CKK 2011].

Theorem

Let G_i have degeneracy k_i and maximum degree $\Delta_i = \Delta_i(G)$. Further, let $k = k_1 + k_2$ and $M = (2\Delta_1 k_1 - \Delta_1 - k_1^2) + (2\Delta_2 k_2 - \Delta_2 - k_2^2) + k_2\Delta_1$. Then $\tau_2(G_1 \times G_2) \le 2k + \left\lceil \frac{1+\sqrt{9+8M}}{2} \right\rceil$.

Proof.

Number the vertices of G_i in increasing order according to a construction sequence. Number the vertices of $G_1 \times G_2$ lexicographically and arrange them in a grid. Consider a construction sequence of increasing lexicographic order. Color $G_1 \times G_2$ with this construction sequence, so each vertex has at most k neighbors which exclude at most $2k$ colors. When colored, a vertex v has at most

$$\left(2\Delta_1 k_1 - \Delta_1 - k_1^2\right) + \left(2\Delta_2 k_2 - \Delta_2 - k_2^2\right) + k_2\Delta_1 = M$$

second-neighbors already colored by an earlier theorem. The first term refers to second-neighbors in the same column, the second to second-neighbors in the same row, and the third to second-neighbors which have neither row or column in common. Thus we need r extra colors, where $\binom{r}{2} \geq M + 1$. Solving, we find $r \geq \frac{1+\sqrt{9+8M}}{2}$. □

Theorem

Let graphs G_1 and G_2 have maximum degrees $\Delta_1 \geq \Delta_2 \geq 1$, $\Delta_1 \geq 2$, and let $\Delta = \Delta(G_1 \times G_2) = \Delta_1 + \Delta_2$. Then

$$\tau_2(G_1 \times G_2) \leq 2\Delta - 1 + \left\lceil \frac{1 + \sqrt{9 + 8(\Delta^2 - \Delta - \Delta_2 - \Delta_1\Delta_2)}}{2} \right\rceil .$$

Proof.

We may assume $G_1 \times G_2$ is connected and regular. Number the vertices of G_i in increasing order according to a construction sequence. Number the vertices of $G_1 \times G_2$ lexicographically and arrange them in a grid. Consider a construction sequence of increasing lexicographic order. Delete an edge e incident with the last vertex of this sequence. Now $G_1 \times G_2 - e$ is $\Delta - 1$-degenerate. □

Proof.

Coloring it using a construction sequence, each vertex has at most $\Delta - 1$ neighbors which exclude at most $2(\Delta - 1)$ colors. There are at most $\Delta_1(\Delta_1 - 1) + (\Delta_2 - 1)^2 + \Delta_1\Delta_2 - 1 = (\Delta_1 + \Delta_2)^2 - (\Delta_1 + 2\Delta_2) - \Delta_1\Delta_2 = \Delta^2 - \Delta - \Delta_2 - \Delta_1\Delta_2$ second-neighbors already colored. The first term refers to second-neighbors in the same column, the second to second-neighbors in the same row, and the third to second-neighbors which have neither row or column in common. Thus we need r extra colors, where $\binom{r}{2} \geq \Delta^2 - \Delta - \Delta_2 - \Delta_1\Delta_2 + 1$. Solving, we find $r \geq \frac{1+\sqrt{9+8(\Delta^2-\Delta-\Delta_2-\Delta_1\Delta_2)}}{2}$. Lastly, adding e back requires at most one more color. □

Note that CKK proved the upper bound $\tau_2(G) \le 2\Delta + \left\lceil \frac{1+\sqrt{9+8\Delta(\Delta-1)}}{2} \right\rceil \le \lceil 2+\sqrt{2} \rceil \Delta$. The improvement on this bound is greatest when $\Delta_1\Delta_2$ is largest (for fixed Δ), namely when $\Delta_1 = \Delta_2 = \frac{1}{2}\Delta$. Then $\Delta^2 - \Delta - \Delta_2 - \Delta_1\Delta_2 = \frac{3}{4}\Delta^2 - \frac{3}{2}\Delta = \frac{3}{4}(\Delta-1)^2 - \frac{3}{4}$, so

$$\left\lceil \frac{1+\sqrt{9+8\left(\frac{3}{4}(\Delta-1)^2 - \frac{3}{4}\right)}}{2} \right\rceil \le \left\lceil \frac{1+\sqrt{3+6(\Delta-1)^2}}{2} \right\rceil \le \left\lceil \frac{\sqrt{6}\Delta}{2} \right\rceil = \left\lceil \sqrt{1.5}\Delta \right\rceil$$

and $\tau_2(G_1 \times G_2) \le 2\Delta - 1 + \lceil \sqrt{1.5}\Delta \rceil = \lceil (2+\sqrt{1.5})\Delta \rceil - 1$.

Theorem

Let $m \leq n$. Then

$$\tau_2(K_m \times K_n) = \begin{cases} 6 & m = n = 2 \\ 2n & \text{else} \end{cases}.$$

Proof.

Let $G = K_m \times K_n$, $m \leq n$. Now $K_n \subseteq G$, so $\tau_2(G) \geq 2n$. Now certainly $\tau_2(K_2 \times K_2) = 6$, and $\tau_2(K_1 \times K_2) = 4$. Furthermore, we see $G \subseteq K_n \times K_n$, so the proof reduces to this case.

Now it is well-known that there exist at least two mutually orthogonal Latin squares for all n except 2 and 6. We construct a 2-tone coloring of $K_n \times K_n$ by using numbers 1 to n for the first Latin square and n+1 to 2n for the second Latin square. Now juxtapose them into a Graeco-Latin square. This can be viewed as a 2-tone coloring of $K_n \times K_n$, where each cell is a vertex and each pair of vertices in the same row or column are adjacent.

Proof.

This leaves us to find a 2-tone coloring of $K_6 \times K_6$ with 12 colors. The coloring below, representing the graph as a table, completes the proof.

AB	37	28	59	46	10
90	68	35	7A	1B	24
78	2B	40	13	5A	69
56	0A	17	48	29	3B
34	15	9B	26	70	8A
12	49	6A	0B	38	57

□

Note that a 2-tone coloring exists in this last case even though a Graeco-Latin square does not because we have 66 labels to choose from, rather than 36.

We can use this result to construct bounds on the 2-tone chromatic number of a cartesian product of graphs.

Corollary

Let G and H be nontrivial graphs, not both K_2, with orders r and s, respectively. Then

$$\max\{\tau_2(G), \tau_2(H), 6\} \leq \tau_2(G \times H) \leq \max\{2r, 2s\}.$$

We can also consider products of paths.

Proposition

For the grid $P_m \times P_n$, $m, n \geq 2$, we have $\tau_2(P_m \times P_n) = 6$.

Proof.

Since the grid contains a 4-cycle, its 2-tone chromatic number is at least six. Represent the grid as a lattice in the first quadrant. Tile the grid with the following block.

36	15	24
25	34	16
14	26	35

This defines a 2-tone coloring. □

Corollary

Let i, j be positive integers. We have $\tau_2(C_{3i} \times C_{3j}) = 6$.

Proof.

Take the coordinates in the previous construction mod $3i$ and $3j$, respectively. This defines a 2-tone coloring for the product of cycles. □

These problems suggest the problem of determining $\tau_2(C_i \times C_j)$ for all i and j. We have the following partial results.

Theorem

We have

$$\tau_2(C_3 \times C_i) = \begin{cases} 6 & i = 3, 6, 8, 9, 11, 12,\ i \geq 14 \\ 7 & i = 4, 5, 7 \end{cases}$$

with $i = 10, 13$ *undecided between 6 and 7.*

Proof.

We have already seen the result for i and multiple of 3. The following block works for $i = 8$, and blocks of length 3 and 8 can be made to agree on two consecutive columns, so they can be concatenated to obtain the other values.

14	56	24	15	36	14	56	23
25	34	16	23	45	26	13	46
36	12	35	46	12	35	24	15

For $i = 4, 5, 7$, the appropriate labelings are easily determined, and the first is included in the following theorem. Suppose that $G = C_3 \times C_4$ has a 2-tone 6-coloring. Then each color is used four times, since 24 non-distinct colors are needed and every color appears in each column. Then each color appears twice in some row, necessarily distance two apart. Thus there is a pair of columns distance two apart with three such pairs of colors in the same rows. □

Proof.

Clearly no two can be in the same row. This trio of colors must be rotated in the column in between. The trio of the other three colors must be rotated to all three positions over these three columns. But this leads to a contradiction.

For $G = C_3 \times C_5$, each color is used five times, and so occurs twice in two rows distance two apart. Thus there are twelve such pairs, so of the five pairs of columns distance two apart, one must have at least three such pairs of colors in the same rows. A contradiction follows as before.

For $G = C_3 \times C_7$, it is possible to show by an exhaustive search that there is no 6-coloring. □

Corollary

For $j = 3, 6, 8, 9, 11, 12,$ *and* $j \geq 14$, $\tau_2(C_{3i} \times C_j) = 6$.

Cartesian Products of Cycles

Lemma

If a product of cycles has a 2-tone 6-coloring, then no vertex has both its two neighbors in its row sharing a common color and its two neighbors in its column sharing a common color.

Proof.

Let v be a vertex in a product of cycles G. Suppose to the contrary that G has a 2-tone 6-coloring and v has both its two neighbors in its row sharing a common color and its two neighbors in its column sharing a common color.

Let u and w be neighbors of v that are not in the same row or column, and let their mutual neighbor other than v be x. If u and w have no colors in common, then x is forced to have the same colors as v, which is impossible. Thus every pair of neighbors of v has a color in common, and clearly no two neighbors of v have the same label.

Form a graph H whose vertices are the four colors not used on v, and edges represent the pairs of colors used to label them. Then H has size four, and every pair of edges is adjacent. But this is impossible.

Theorem

Let $i \geq 3$. Then $\tau_2(C_4 \times C_i) = 7$.

Proof.

Consider a given row of $G = C_4 \times C_i$. It is possible for a row to be 2-tone colored with six colors so that every vertex has the property that each two neighbors share no common colors $\iff$ its length is a multiple of three.
First suppose $i \neq 0$ mod 3 and the graph G has a 2-tone 6-coloring. Now every 4-cycle 2-tone colored with six colors has every pair of nonadjacent vertices having one common color. Thus G has some vertex v so that its pairs of neighbors in both its row and column have one color in common. But by the previous lemma, this is impossible. □

Proof.

Now suppose $i = 0 \mod 3$, and G has a 2-tone 6-coloring. By the lemma, each row must have the property that each two neighbors share no common colors. Then each six colors appear in every three consecutive vertices of a row, and two nonconsecutive rows must have a color repeated in each of the columns. Then a contradiction follows as in the case of the graph $G = C_3 \times C_4$. Thus $\tau_2(C_4 \times C_i) \geq 7$.
We now show that equality can be achieved. The following tables correspond to colorings of $C_4 \times C_3$. and $C_4 \times C_4$. Each entry represents a vertex and vertices are adjacent if they are adjacent in a row or column, or on opposite ends of a row or column.

15	46	23
37	25	14
12	36	57
34	27	16

15	46	12	47
37	25	34	26
12	36	17	45
34	27	35	16

15	46	12	47	36
37	25	34	26	14
12	36	17	45	37
34	27	35	16	25

These two colorings can be concatenated to form products of C_4 and larger cycles since they agree on the first two columns. This leaves only $G = C_4 \times C_5$, which is shown above.

Lemma

Let $n \geq 5$, $n \neq 0$ mod 3. Then any 2-tone 6-coloring of C_n uses at least four 2-chords.

Proof.

Case 1. Let $n = 3r+1$. Then C_n needs $6r+2$ colors with repetition. A color class using no 2-chords can use at most r vertices. A color class with $r+1$ vertices uses at least two 2-chords. Four classes of r vertices and two of $r+1$ produce $6r+2$ colors. Using a color class of $r+k$ colors would require at least $3k-1$ 2-chords, so there is no advantage to using a class of more than $r+1$ vertices.
Case 2. Let $n = 3r+2$. Then C_n needs $6r+4$ colors with repetition. A color class using no 2-chords can use at most r vertices. A color class with $r+1$ vertices uses at least one 2-chord. Two classes of r vertices and four of $r+1$ produce $6r+4$ colors. Using a color class of $r+k$ colors would require at least $3k-2$ 2-chords, so there is no advantage to using a class of more than $r+1$ vertices.
In both cases, at least four 2-chords are required. □

Cartesian Products of Cycles

Proposition

Let $(i,j) = (5,n)$, $5 \leq n \leq 19$, $n \neq 0$ *mod 3, or* $(7,7)$ *or* $(7,8)$. *Then* $\tau_2(C_i \times C_j) \geq 7$.

Proof.

Each 2-chord corresponds to the vertex between its ends. In each case, we show that if the graph has a 2-tone 6-coloring, more 2-chords are required than there are vertices in the product of cycles. Then by the pigeonhole principle, some vertex must use 2-chords in both its row and column. But by the earlier lemma, this is impossible, so the 2-tone chromatic number is at least 7.

For $(i,j) = (5,n)$, $5 \leq n \leq 19$, $n \neq 0$ mod 3, we have order $5n$ and at least $4n + 4 \cdot 5 > 5n$ 2-chords. For $(i,j) = (7,7)$, we have order 49 and at least $4 \cdot 7 + 4 \cdot 7 = 56$ 2-chords. For $(i,j) = (7,8)$, we have order 56 and at least $4 \cdot 7 + 4 \cdot 8 = 60$ 2-chords. □

We can also consider 2-tone coloring for the hypercubes. This appears difficult in general, but we do have the following result.

Corollary

We have the following values for the 2-tone chromatic number for hypercubes.

n	$\tau_2(Q_n)$
1	4
2	6
3	6
4	7

Proof.

We have $Q_1 = K_2$, $Q_2 = C_4$, $Q_3 = C_4 \times K_2$, and $Q_4 = C_4 \times C_4$, so these values have already been determined. □

The 3-cube can be colored with six colors by making its color classes its two partite sets and four pairs of vertices distance three apart. It is not hard to verify that this coloring is unique up to isomorphism of colors and graphs. The 4-cube contains the 3-cube, but attempting to extend this 6-coloring quickly leads to a contradiction.

The 7-coloring used for the 4-cube does not extend to the 5-cube, but that does not prove that it cannot be 7-colored. It was shown in the original report by Josh Goss that the 5- and 6-cubes can be 8-colored.

Thank You!

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