New Results on 2-Tone Coloring

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History of t-Tone Coloring

- Math 645 (Summer 2009) taught by Ping Zhang - Class project of Nicole Fonger, Josh Goss, Ben Phillips, and Chris Segroves (FGPS) - assisted by Gary Chartrand
- Bickle and Phillips work on the topic Summer 2009-2011 - submit paper *"t-Tone Colorings of Graphs"* (2011)
- Bickle presents at conferences - 41st Southeastern (2010) and MIGHTY LI (2011) - attended by Doug West
- Cranston, Kim, and Kinnersly (CKK), current and former students of West, submit the paper *"New Results in t-Tone Coloring* of Graphs" following up on our work (2011)
- Thanks to Drs. Gary Chartrand, Allen Schwenk, Doug West, and Ping Zhang for their advice

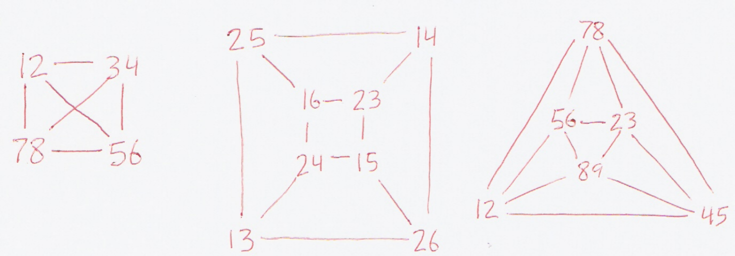
Definition

A 2-tone coloring of a graph assigns two colors to each vertex of a graph so that adjacent vertices have no common colors and vertices at distance two have at most one common color.

The label of a vertex is the pair of colors on a vertex.

A graph is 2-tone k-colorable if it can be 2-tone colored with k colors.

The 2-tone chromatic number $\tau_2(G)$ of a graph is the minimum number of colors in any 2-tone coloring.



Clearly $\tau_2(K_n) = 2n$.
Thus $2\omega(G) \leq \tau_2(G) \leq 2n$.
Also, $\tau_2(G) \geq \frac{2n}{\alpha(G)}$.

Theorem

[FGPS] We have $\tau_2(G) \leq \chi(G) + \chi(G^2)$.

Proof.

Combining a proper vertex coloring of G and a proper coloring of G^2 yields a 2-tone coloring of G. $\square$

Complete Multipartite Graphs

Theorem

[FGPS] For the complete multipartite graph $K_{a_1,a_2,\ldots,a_r}$,

$$\tau_2\left(K_{a_1,\ldots,a_r}\right) = \sum_{i=1}^{r} \left\lceil \frac{1+\sqrt{1+8a_i}}{2} \right\rceil .$$

Proof.

Each partite set cannot use any color in common with any other. In a partite set, we need need r colors, where $\binom{r}{2} \geq a_i$. Solving for r gives the formula. $\square$

Corollary

[FGPS] For the nontrivial star $K_{1,s}$,

$$\tau_2(K_{1,s}) = \left\lceil \frac{5+\sqrt{1+8s}}{2} \right\rceil.$$

Proof.

$$\tau_2(K_{1,s}) = 2 + \left\lceil \frac{1+\sqrt{1+8s}}{2} \right\rceil = \left\lceil \frac{5+\sqrt{1+8s}}{2} \right\rceil.$$

□

Corollary

Let G have maximum degree $\triangle$. Then

$$\tau_2(G) \geq \left\lceil \frac{5+\sqrt{1+8\triangle}}{2} \right\rceil.$$

Bridges

Theorem

Let G be a graph with a bridge $e = uv$. Let F_1 and F_2 be the components of $G - e$ containing u and v, respectively, and let $H_1 = G[F_1 \cup v]$, and $H_2 = G[F_2 \cup u]$. Then $\tau_2(G) = \max\{\tau_2(H_1), \tau_2(H_2)\}$.

Proof.

The two subgraphs can be colored to agree on the bridge without conflict. □

Theorem

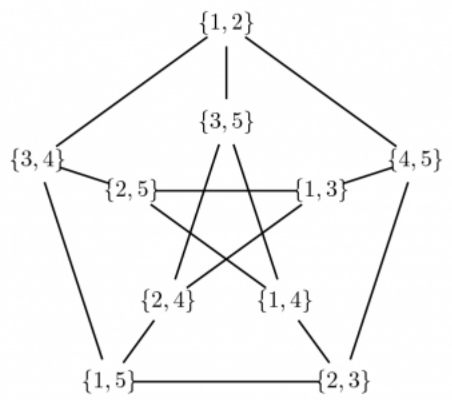
[FGPS] Let T be a nontrivial tree with maximum degree $\triangle$. Then

$$\tau_2(T) = \left\lceil \frac{5 + \sqrt{1 + 8\triangle}}{2} \right\rceil.$$

Proof.

[Allan] By the theorem on bridges, the 2-tone chromatic number of a tree is the maximum of the 2-tone chromatic numbers of all the stars it contains. □

Consider forming a graph whose vertices are the ten possible labels for a 2-tone 5-coloring and all possible edges are added. This graph is the Petersen graph. In fact, the Petersen graph can be defined with just this labeling. Thus its 2-tone chromatic number is five, so any subgraph of the Petersen graph is 2-tone 5-colorable.



The Petersen Graph

Theorem

If a graph G has diameter at most four, then G is 2-tone 5-colorable $\iff$ G is a subgraph of the Petersen graph.

Proof.

We have seen that every subgraph of the Petersen graph is 2-tone 5-colorable.

If G is 2-tone 5-colorable, it is easily seen that two vertices with the same label must be distance at least five apart, since otherwise $C_4 = K_{2,2}$ could be colored with five colors. Thus if G has diameter at most four, then no label can be repeated, so it is a subgraph of the Petersen graph. $\square$

Definition

A 2-chord of a cycle to be a pair of vertices of the cycle whose distance is two.

Theorem

The Petersen Graph has no 7-cycle.

Proof.

Suppose it has a 7-cycle. Then C_7 has seven 2-chords, and each must have at most one common color. Now 14 colors are used with repetition, and each color can be used at most three times. Then four colors are used three times, and one is used twice. But this implies that C_7 has at least eight 2-chords. $\square$

Theorem

[FGPS] For the cycle C_n,

$$\tau_2(C_n) = \begin{cases} 6 & n = 3, 4, 7 \\ 5 & else \end{cases}.$$

Proof.

[Allan] Certainly $\tau_2(C_n) \geq 5$. Now the cycles of length 3, 4, and 7 are not subgraphs of the Petersen graph. Now $C_3 = K_3$, and C_4 can have nonadjacent vertices share a color, so $\tau_2(C_3) = \tau_2(C_4) = 6$. Now C_7 can be labeled as below. The cycles of length 5, 6, 8, and 9 are subgraphs of the Petersen graph, with labellings below represented as broken cycles. □

Proof.

$$-12-34-51-23-45-$$

$$-12-34-15-32-14-35-$$

$$-12-34-56-13-24-35-46-$$

$$-12-34-15-23-14-25-13-45-$$

$$-12-34-15-32-14-25-13-24-35-$$

Finally, for $n \geq 10$, the cycle can be constructed by breaking and attaching together cycles of length 5, 6, 8, and 9, which can be done because the labellings above agree on the first three vertices. $\square$

Regular Graphs

Theorem

If G is r-regular, $r \geq 2$, then $\tau_2(G) \leq r^2 + r$.

Proof.

Let G be r-regular, $r \geq 2$. Note that each vertex of G has at most $r + r(r-1) = r^2$ other vertices within distance two, so $\triangle(G^2) \leq r^2$. Hence

$$\tau_2(G) \leq \chi(G) + \chi(G^2) \leq (1 + \triangle(G)) + (1 + \triangle(G^2)) \leq (1+r) + (1+r^2).$$

By Brooks' Theorem, the middle inequality can be an equality only when G or G^2 are complete or an odd cycle. Now for $r \geq 2$, $\tau_2(K_r) = 2r \leq r^2 + r$ and $\tau_2(C_n) \leq 6 \leq r^2 + r$. Now G^2 cannot be a noncomplete odd cycle. The only case in which we might have $\chi(G^2) = (1 + r^2)$ is when G^2 is complete with order $1 + r^2$. □

Proof.

But this means that G must have diameter 2 and girth 5, so G is a Moore graph. But Hoffman and Singleton [1960] showed that the only Moore graphs with girth 5 occur when $r = 2, 3, 7$, and possibly 57 (this case is undecided). The case $r = 3$ is the Petersen graph (PG), which has $\tau_2(PG) = 5 < 12$. The case $r = 7$ is the Hoffman-Singleton graph (HS), for which $\chi(HS) = 4$, so $\tau_2(HS) \leq 4 + 49 < 56$. Finally, Borodin and Kostochka [1977] showed that for a K_4-free graph, $\chi(G) \leq \left\lceil \frac{3}{4}(\triangle(G)+1) \right\rceil$, so if a graph M satisfies the final case $r = 57$, then $\chi(M) \leq 44$, so $\tau_2(M) \leq 44 + 57^2 < 57 + 57^2$. □

This implies that for any graph G, $\tau_2(G) \leq \Delta^2 + \Delta$.

Conjecture

Let G have maximum degree $\triangle$. Then $\tau_2(G) \leq 2\triangle + 2$, with equality only if G contains $K_{\triangle+1}$ or for $\triangle = 2$, C_4 or C_7.

Critical Graphs

Definition

A graph G is 2-tone k-critical if $\tau_2(G) = k$ and for any proper subgraph H of G, $\tau_2(H) < k$.

Theorem

Let G be a graph containing P_3 with edge $e = uv$. Then $\tau_2(G) - \tau_2(G - e) \leq 1$.

The proof breaks into three cases depending on whether the labels of u and v have 0, 1, or 2 colors in common. The case when the labels are the same is the most difficult.

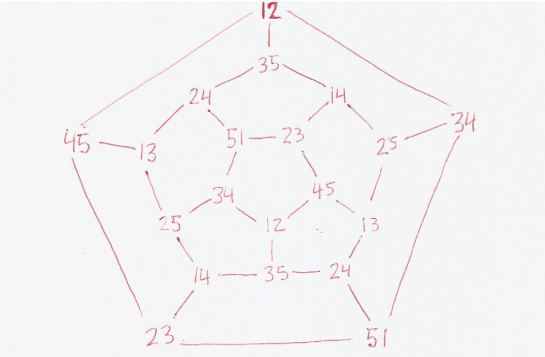
We can characterize the 6-critical graphs in one particular family of graphs. The theta graph $\theta_{i,j,k}$ is formed by taking paths of lengths i, j, k and identifying them at their end-vertices. It necessarily contains three cycles of lengths $a = i + j$, $b = i + k$, $c = j + k$. We will use (i,j,k) for $\theta_{i,j,k}$.

Theorem

The theta graph $\theta_{1,2,2}$ has $\tau_2(\theta_{1,2,2}) = 7$, and $(3,3,3)$, $(3,3,5)$, $(3,3,6)$, $(4,4,4)$, $(4,4,5)$, and $(3,3,9)$, all have 2-tone chromatic number 6. For all other theta graphs, $\tau_2(\theta_{i,j,k}) = \max\{\tau_2(C_a), \tau_2(C_b), \tau_2(C_c)\}$, where $a = i + j$, $b = i + k$, $c = j + k$.

Definition

A Petersen cover is a covering graph of the Petersen graph. That is, it is a graph for which there is an onto homomorphism f from G to the Petersen graph with the property that for each vertex v of G the neighborhood of v maps bijectively onto the neighborhood of $f(v)$.



Theorem

A cubic graph G has $\tau_2(G) = 5$ if and only if G is a Petersen cover.

Proof.

Let G be a cubic graph G with $\tau_2(G) = 5$. Then given a 2-tone 5-coloring of G, map all the vertices with the same label to the vertex of the Petersen graph with that label. Now no edge of G is mapped to a nonedge of the Petersen graph, since then it would violate the labeling. No adjacent edges are mapped to the same edge, since then there would be two vertices at distance two with the same label. Thus G is a Petersen cover.

Let G be a Petersen cover. Label all the vertices that map to a given vertex of the Petersen graph under the homomorphism with the same label. This produces a 2-tone coloring of G. $\square$

Corollary

A Petersen cover has order $n = 10k$, k a positive integer.

Proof.

Let G be a Petersen cover. Suppose k vertices receive label X in a 2-tone 5-coloring, and one of them is v. Then every other label appears exactly once amongst the vertices within distance two of v, as in the Petersen graph. Further, none of these vertices is within distance two of any other vertex with label X, since such vertices must be distance at least five apart. By considering another label, we see that the number of vertices receiving each label is the same. Thus the order of G is a multiple of ten. □

Corollary

If G is cubic, with order $n \neq 10k$, it is not 2-tone 5-colorable.

Definition

Let G be a graph containing vertices u, v, w, x, edges uv, wx, and not containing edges uw, vx. A 2-switch is the operation that deletes edges uv and wx and adds edges uw and vx.

Theorem

G is a Petersen cover if and only if it can be obtained by starting with k disjoint copies of the Petersen graph and performing some number of 2-switches on pairs of edges that join vertices with the same labels.

Proof.

Consider a graph produced by this process. The edges switched by a 2-switch must still map to the same edge of the Petersen graph, so the graph is a Petersen cover.

Let G be a Petersen cover. Consider aligning the vertices that map to the same vertex of the Petersen graph in a column of k levels, with one vertex of each type per level. By performing 2-switches, it is possible to produce edges that all join vertices of the same level, separating out k copies of the Petersen graph. Reversing the sequence of 2-switches, we see that G can be constructed in this way. $\square$

Theorem

Let G be a Petersen cover. Then
1. G does not contain a bridge.
2. If G contains a minimal 2-edge-cut, then performing a 2-switch on those edges separates G into two components which are two smaller Petersen covers.
3. If G has a minimal 3-edge-cut, it is trivial.

Proof.

(1 only) Performing a 2-switch on a pair of edges, of which at least one is a bridge produces another pair of edges, at least one of which is a bridge. Thus no bridge can be produced by the process described in the previous theorem. □

Cubic Graphs

In 2010, I made the following conjecture for cubic graphs.

Conjecture

Let G be a cubic graph. Then
1. $\tau_2(G) \leq 8$;
2. $\tau_2(G) \leq 7$ when G does not contain K_4;
3. $\tau_2(G) \leq 6$ when G does not contain $K_4 - e$.

I had verified this conjecture for $C_n \times K_2$, mobius ladders, cubic graphs of order at most 8, and various others.

CKK improved on our general upper bound. Their key idea is to color greedily using the available labels, rather than combining two separate colorings.

If you color a graph with maximum degree Δ, then the neighbors of a vertex exclude 2Δ colors. There are at most $\Delta(\Delta-1)$ "second-neighbors" of a vertex. Since $\binom{\sqrt{2}\Delta}{2} > \Delta(\Delta-1)$, we have $\tau_2(G) \leq \lceil 2+\sqrt{2} \rceil \Delta$.

Theorem

[CKK 2011] We have $\tau_2(G) \leq \lceil 2+\sqrt{2} \rceil \Delta$.

This is not quite the best possible, as CKK chose a simpler formula over exactness. It is also possible to shave one off this bound. We first consider a bound based on maximum degree and degeneracy.

Theorem

[Allan 2011] Let G be k-degenerate with maximum degree $\Delta = \Delta(G)$*. Then* $\tau_2(G) \leq 2k + \left\lceil \frac{1+\sqrt{9+8(2\Delta k-\Delta-k^2)}}{2} \right\rceil$.

Proof.

Color G with a construction sequence. Each vertex has at most k neighbors which exclude at most $2k$ colors. When colored, a vertex v has $j \leq k$ neighbors already colored and each of these has at most $\Delta - 1$ second-neighbors. Then v has at most $\Delta - j$ uncolored neighbors, each of which has at most $k-1$ colored second-neighbors. Thus there are at most $k(\Delta-1)+(\Delta-k)(k-1) = 2\Delta k - \Delta - k^2$ second-neighbors already colored, so we need r extra colors, where $\binom{r}{2} \geq 2\Delta k - \Delta - k^2 + 1$. Solving, we find $r \geq \frac{1+\sqrt{9+8(2\Delta k-\Delta-k^2)}}{2}$. □

Theorem

Let a nonempty graph G have maximum degree $\Delta = \Delta(G)$. Then $\tau_2(G) \leq 2\Delta - 1 + \left\lceil \frac{1+\sqrt{1+8\Delta(\Delta-1)}}{2} \right\rceil$.

Proof.

We may assume G is connected and regular. Now $G-e$ is $\Delta-1$-degenerate. Coloring it using a construction sequence, each vertex has at most $\Delta-1$ neighbors which exclude at most $2(\Delta-1)$ colors. There are at most $\Delta(\Delta-1)-1$ second-neighbors already colored, so we need r extra colors, where $\binom{r}{2} \geq \Delta(\Delta-1)$. Solving, we find $r \geq \frac{1+\sqrt{1+8\Delta(\Delta-1)}}{2}$. Lastly, adding e back requires at most one more color by the theorem on critical graphs. □

Note that the bound in the theorem is at least one better than that of CKK.

Upper Bounds

Theorem

[CKK 2011] Let a nonempty bipartite graph G have maximum degree $\Delta = \Delta(G)$. Then $\tau_2(G) \le 2\left\lceil \sqrt{2\Delta} \right\rceil$.

Proof.

Use disjoint sets of colors for the partite sets. Coloring a partite set using a construction sequence, each vertex has no neighbors in the same partite set and has at most $\Delta(\Delta-1)$ second-neighbors, so we need r colors, where $\binom{r}{2} > \Delta(\Delta-1)$. The bound follows. □

Using similar techniques, I proved a bound which can be one less, (e.g. $\Delta \in \{3, 5, 10, 15, 17, 22, 27, 29\}$), but usually is one more.

Theorem

[Allan 2011] Let a nonempty bipartite graph G have maximum degree $\Delta = \Delta(G)$. Then $\tau_2(G) \le 2\left\lceil \frac{1+\sqrt{1+8\Delta(\Delta-1)}}{2} \right\rceil + 1$.

CKK proved analogous bounds for chordal graphs.

Theorem

[CKK 2011] a. If G is a chordal graph, then
$\tau_2(G) \le \left\lceil \left(1 + \frac{\sqrt{6}}{2}\right) \Delta \right\rceil + 1$.
b. For every $\varepsilon > 0$, there exists an r_0 such that whenever $r > r_0$, if G is a chordal graph with maximum degree r, then
$\tau_2(G) \le (2 + \varepsilon) r$.

Thus for cubic graphs, we have

- $\tau_2(G) \leq 12$ (our original bound)
- $\tau_2(G) \leq 11$ (CKK's general Δ bound)
- $\tau_2(G) \leq 9$ (my improvement of CKK's bound)
- $\tau_2(G) \leq 8$ (CKK's main theorem)

Theorem

[CKK 2011] If $\Delta(G) \leq 3$, then $\tau_2(G) \leq 8$.

The proof is about three pages long. I will provide a brief sketch.

Cubic Graphs

Proof sketch:

- A 2-degenerate graph with $\Delta = 3$ is 2-tone 8-colorable.
- Let G be a counterexample of minimum order.
- G must be cubic, since else it is 2-degenerate and can be colored as above.
- G cannot have an induced $K_{2,3}$, since if it did, the coloring can be extended to it.
- Color all of G except for a cycle C of minimum length.
- Either G contains $K_4 - e$, or none of the vertices of C have a common neighbor.
- There are three cases, depending upon the specific partial coloring.
- We can color the vertices along the cycle, maintaining 'flexibility', and complete the coloring.

How about other cubic graphs? Recall part 3 of my conjecture:

Conjecture

Let G be a cubic graph. Then $\tau_2(G) \leq 6$ when G does not contain $K_4 - e$.

CKK refuted this by demonstrating that it fails for the Heawood graph, which does not contain $K_4 - e$ (and indeed has girth 6). Their proof uses a clever but somewhat involved contradiction. I developed my own shorter proof which analyzes the maximal independent sets of the Heawood graph.

Theorem

[CKK 2011] The Heawood Graph is not 2-tone 6-colorable.

Proof.

[Allan 2011] Recall that the Heawood Graph G is the incidence graph for the Fano Plane, or equivalently, the Steiner Triple System of order 7 (STS(7)). Hence it is bipartite and any two vertices in the same partite set have exactly one common neighbor.

Consider the maximal independent sets of G. Each partite set is a maximum independent set of size seven. A vertex in one partite set is adjacent to three in the other, so there can be four vertices from the other partite set in the maximal independent set. Call an independent set with four vertices in one partite set and one in the other a 1-4-set. Two vertices in one partite set have a total of five distinct neighbors in the other set. Hence there can be an independent set with two vertices in each partite set. □

Proof.

Suppose that G is 2-tone 6-colorable. If there is a color class of size seven, then it is one of the partite sets, and each vertex requires a distinct color for its other color. If there is a color class of size six contained in one of the partite sets, then there are at most two color classes with two vertices in this partite set, and at least four more colors are needed.

Hence each color class has size at most five. But then the six color classes must have sizes 5,5,5,5,4,4. Restricted to one partite set, they have sizes 5, 4, 2, or 1. Hence there are at least two with size at least four. Now any two color classes can overlap on at most one vertex of a partite set, so there are two with size four, which must 1-4-sets.

Now WLOG any 1-4-set contains four vertices that do not contain a triangle of the STS(7). But no other 1-4-set can contain only one vertex of this set, since then it would contain a triangle. This is a contradiction. □

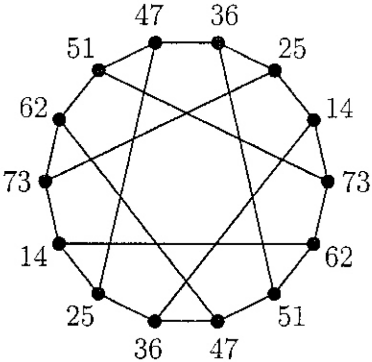


Fig. 1: A 2-tone 7-coloring of the Heawood graph.

After learning of the falsification of part 3, I checked all 21 cubic graphs of order 10 and found that one of them also violates part 3.

Theorem

[Allan 2011] Let G be the graph formed by starting with two copies of $K_{2,3}$ and adding a matching between the vertices of degree two in the two $K_{2,3}$'s. Then G is 2-tone 7-critical.

Proof.

Suppose G has a 2-tone 6-coloring. Now $K_{2,3}$ is uniquely 6-colorable, and each partite set requires three distinct colors. Let A and B be the partite sets of one $K_{2,3}$ and C and D be the partite sets of the other $K_{2,3}$, with A and C being those that have three vertices, and hence are joined by the matching.

If there is a color in common between A and C, then it must appear on two vertices in each set, and hence on adjacent vertices. If they have no common colors, then B and C use the same three colors. But then they have two common labels at distance two apart.

G has only two edge orbits. Considering $G - e$ for one edge of each type, 2-tone 6-colorings are easily obtained. Hence G is 7-critical. $\square$

These two graphs are the only known counterexamples. It is interesting that they are both bipartite.

Definition

A t-tone coloring of a graph assigns t colors to each vertex of a graph so that vertices at distance d have fewer than d common colors.
The t-tone chromatic number $\tau_t(G)$ of a graph is the minimum number of colors in any t-tone coloring.

CKK proved the following four theorems for t-tone coloring.

Theorem

We have $\tau_t(G) \leq (t^2 + t)\Delta(G)$.

t-Tone Coloring

Theorem

For all $t > 0$, there exists $c = c(t)$ such that for every tree T we have $\tau_t(T) \leq c\sqrt{\Delta(T)}$.

Theorem

If G is a k-degenerate graph, $k \geq 2$, and $\Delta(G) \leq r$, then for every t we have $\tau_t(G) \leq kt + kt^2 r^{1-\frac{1}{t}}$.

Theorem

For each $r \geq 3$, there exists a constant c such that for all t, there is a graph G for which $\Delta(G) = r$ and $\tau_t(G) \geq \frac{ct^2}{\lg t}$.

Theorem

For all $t > 0$, *there exists* $c = c(t)$ *such that for every tree* T *we have* $\tau_t(T) \leq c\sqrt{\Delta(T)}$.

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t-Tone Coloring

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Thank You!

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