Bounds and Algorithms for Graph Coloring

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A Scheduling Problem

Example. Six math students will take seven summer math classes, denoted A-G. The students' schedules are listed below.

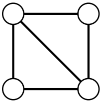
Student	Classes
Al	A, D, G
Bob	B, E
Carl	A, C, G
Dave	C, E
Edna	E, F
Frank	B, D, F

- We model this situation with a graph.

Defining a Graph

Definition

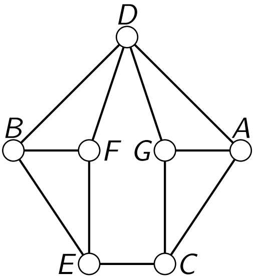
A **graph** G is a mathematical object consisting of a finite nonempty set of objects called **vertices** $V(G)$, and a set of **edges** $E(G)$.
An **edge** is two-element subset of the vertex set.
The **order** $n(G) = |V(G)|$ of a graph G is the number of vertices of G.
The **size** $m(G) = |E(G)|$ of a graph G is the number of edges of G.



$$n = 4$$
$$m = 5$$

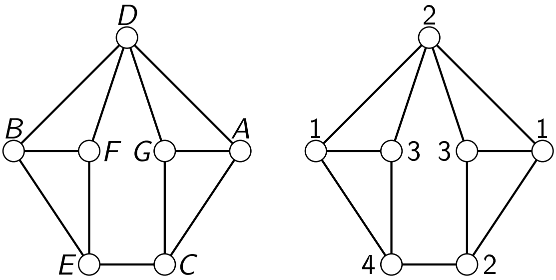
Solving the Example

Student	Classes
Al	A, D, G
Bob	B, E
Carl	A, C, G
Dave	C, E
Edna	E, F
Frank	B, D, F



- Each vertex represents a class. Put an edge between two classes when they have a common student.
- Thus Al's schedule imposes edges AD, AG, and DG.
- Thus we construct the graph above, the **Moser spindle**.
- A natural question to ask here is how few time slots we can schedule the classes in, and how to construct such a schedule.

Solving the Example



- If a vertex is in one slot, its neighbors must be in different slots. Number the slots 1, 2, 3, ...
- A, D, and G must be in three different slots, say 1, 2, and 3.
- If we try to schedule the rest of the classes in three slots, C must be in slot 2.
- Vertices B and F must use slots 1 and 3.
- However, we find it is not possible to schedule vertex E in slots 1, 2, or 3, so a fourth slot is needed.

Other Applications

- Many other situations can be modeled similarly

Situation	Vertices	Minimize
conflicting meetings	meetings	number of time slots
traffic intersection	traffic lanes	cycles of traffic light
conflict in fish tanks	fish	number of tanks
conflict of TV broadcasts	TV stations	number of channels
shipping chemicals	chemicals	number of packages

- In the previous example, we refer to the slots as **colors**, and consider coloring the vertices.

Defining Graph Coloring

Definition

A **vertex coloring** of a graph assigns one color to each vertex.
A **proper vertex coloring** requires that adjacent vertices are colored differently.

- While the colors could be actual colors (red, green, blue, ...), it is common to use natural numbers 1, 2, ... , k.

Definition

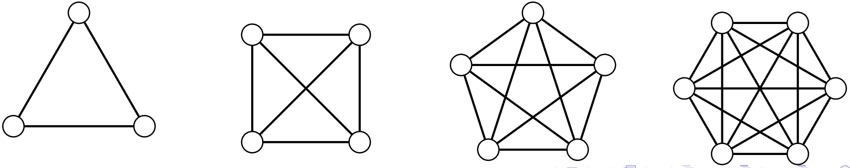
A k-**coloring** of a graph is a proper vertex coloring using **colors** 1, ... , k (not necessarily all of them).
A graph is k-**colorable** if it has a k-coloring.
The **chromatic number** $\chi(G)$ is the minimum number of colors used in any k-coloring of a graph G.
A **minimum coloring** of a graph is one using $\chi(G)$ colors.
A **color class** is all vertices with the same color in some coloring of the graph.

Basic Bounds on Chromatic Number

- To determine the chromatic number of a graph, it is useful to have bounds that are easier to calculate. It is immediate that

$$1 \leq \chi(G) \leq n(G)$$

- An **extremal graph** is one that makes a bound an equality.
- The extremal graph for the lower bound is the **empty graph** $\overline{K}_n$ (which has no edges), since only a graph with no edges can be colored with one color.
- The extremal graph for the upper bound is the **complete graph** K_n (which has all possible edges), since only this requires a different color on each vertex.



- To determine chromatic numbers exactly, we need better bounds.

Definition

A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ so that the edges in $E(H)$ use only vertices in $V(H)$.
We write $H \subseteq G$ and say G **contains** H.
A **clique** is a complete subgraph, or the set of vertices inducing a complete subgraph.
An **independent set** of vertices is a set that induces no edges.

Proposition

If $H \subseteq G$, then $\chi(H) \leq \chi(G)$.

Proof.

A coloring of G with $\chi(G)$ colors can be restricted to H. □

Clique Number and Independence Number

Definition

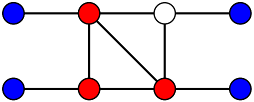
The **clique number** $\omega(G)$ of a graph G is the size of the largest clique of G.

Corollary

For any graph G, $\chi(G) \geq \omega(G)$.

Definition

The **independence number** $\alpha(G)$ of a graph G is the size of the largest independent set of G.



$\omega = 3$
$\alpha = 4$

- The independence number and clique number are complementary parameters.

Lower Bounds

- Any color class in a proper vertex coloring is an independent set.
- A k-coloring partitions the vertex set into k color classes.
- The chromatic number is the smallest number of independent sets into which $V(G)$ can be partitioned.

Proposition

For any graph G, $\chi(G) \geq \frac{n}{\alpha(G)}$.

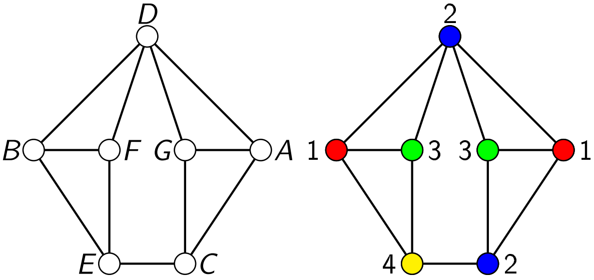
Proof.

Let $k = \chi(G)$, so G has color classes $V_1, \ldots, V_k$ for some k-coloring. Then $n = \sum_{i=1}^{k} |V_i| \leq k \cdot \alpha(G)$. Thus $\chi(G) \geq \frac{n}{\alpha(G)}$. $\square$

- The two basic lower bounds on the chromatic number are $\omega(G)$ and $\frac{n}{\alpha(G)}$.
- Which is better depends on the graph.

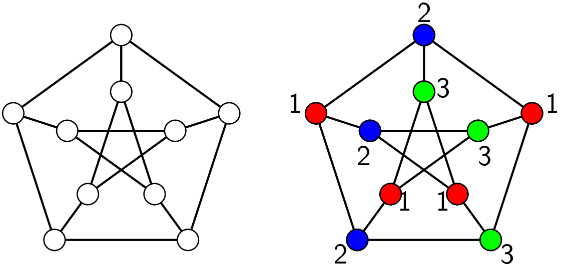
Examples

Example. Find ω, α, and χ for the following graphs.



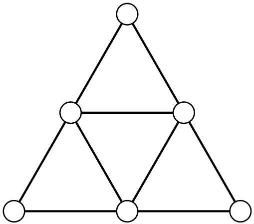
- The Moser spindle has $\omega(G) = 3$, and $\alpha(G) = 2$.
- Corollary 7 implies $\chi(G) \geq 3$, and Proposition 9 implies $\chi(G) \geq \frac{7}{2} = 3.5$.
- Since the chromatic number must be an integer, $\chi(G) \geq 4$.
- A 4-coloring is shown at right, so $\chi(G) = 4$.

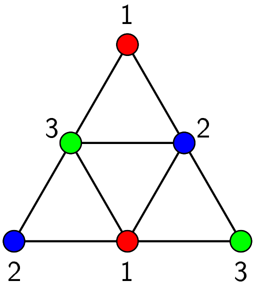
Examples



- The **Petersen graph** has $\omega(G) = 2$ and $\alpha(G) = 4$, with the four vertices colored 1 below being one maximum independent set.
- The latter implies that $\chi(G) \geq \frac{10}{4} = 2.5$.
- A 3-coloring is shown at right, so $\chi(G) = 3$.

Examples





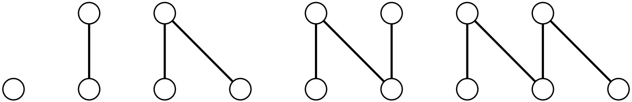
- The graph at right has $\omega(G) = 3$ and $\alpha(G) = 3$, with the three corners being the unique maximum independent set.
- These imply lower bounds of 3 and $\frac{6}{3} = 2$ for the chromatic number.
- A 3-coloring is shown at right.
- Note that the maximum independent set cannot be a color class in any 3-coloring of this graph.

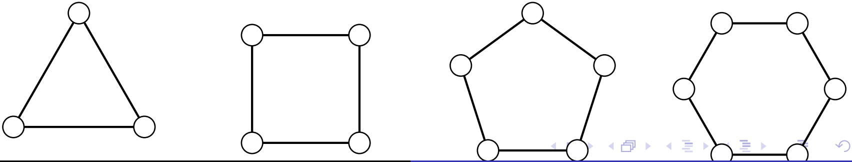
Paths and Cycles

Definition

A **path** P_n is a graph whose vertices can be numbered v_1, v_2, ..., v_n so that its edges are v_1v_2, ..., $v_{n-1}v_n$.
A **cycle** C_n (or n-cycle) is a graph whose vertices can be numbered v_1, v_2, ..., v_n so that its edges are v_1v_2, ..., $v_{n-1}v_n$, and v_nv_1.
An **even cycle** has n even and an **odd cycle** has n odd.

- Small paths and cycles are illustrated below.





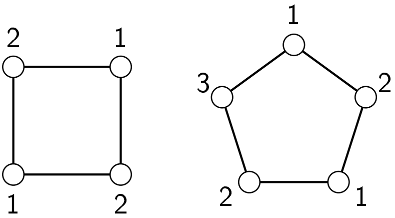
Coloring Cycles

Theorem

A graph is 2-colorable if and only if it contains no odd cycle.

- 2-colorable graphs are also known as bipartite graphs.

Example. Even cycles have $\chi(C_{2k}) = 2$. Odd cycles have $\chi(C_{2k+1}) = 3$.



- We have good characterizations of graphs with chromatic number 1 or 2.

NP-Complete Problems

- Unfortunately, there is no good characterization of graphs with $\chi(G) = k$ when $k \geq 3$.
- In fact, determining $\chi(G)$ when $k \geq 3$ is an NP-complete problem.
- The class of NP-complete problems could all be solved in polynomial time if any can be solved in polynomial time.
- The fact that these problems have been studied extensively without anyone finding a polynomial time solution for any of them suggests (but does not prove) that no such algorithm exists.
- Determining α and ω are also NP-complete problems.
- They are essentially equivalent due to complementation.
- A naive algorithm would check all 2^n vertex subsets of a graph. A better algorithm (Robson [1986]) runs in $\mathcal{O}(1.2108^n)$ time, but no polynomial algorithm is known.

- To show that $\chi(G) = k$, we must show

1. $\chi(G) \geq k$. Use a lower bound, or find a contradiction to show that $\chi(G) < k$ is impossible.
2. $\chi(G) \leq k$. Find a k-coloring, or use an upper bound.

- How can we find a k-coloring?
- Trial and error may work for small graphs, but larger graphs may require a more systematic approach.

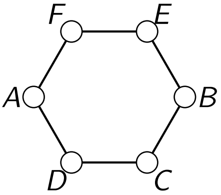
Algorithm

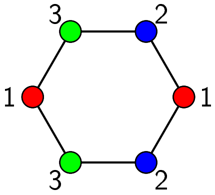
(**Greedy Coloring**) *Given some vertex order, color each vertex with the smallest color that has not already been used on an adjacent vertex.*

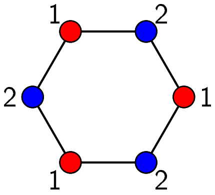
Algorithm

(**Greedy Coloring**) *Given some vertex order, color each vertex with the smallest color that has not already been used on an adjacent vertex.*

Example. Color the vertices of the graph below left in order A-F. The 3-coloring produced is in the center. However, the coloring at right uses only two colors.







Vertex Degrees

- Greedy coloring must produce a proper coloring, but as with many greedy algorithms, it is not guaranteed to produce an optimal solution.
- How good the coloring is depends on the vertex order used.
- Some vertex order must produce a minimum coloring, but checking all $n!$ vertex orders is not practical.
- A better vertex order comes from vertex degrees.

Definition

The **degree** $d(v)$ of a vertex v is the number of edges incident with v.
The **degree sequence** of a graph G is the list of its degrees, usually written in nonincreasing order.
Its **minimum degree** is $\delta(G)$. Its **maximum degree** is $\Delta(G)$.
It is **regular** if every vertex has the same degree (k-**regular** if the common degree is k).

The Maximum Degree Bound

- In any vertex order, each vertex has at most $\Delta(G)$ previously colored neighbors.
- Thus using $1+\Delta(G)$ colors, there is always a color available to color the next vertex.

Theorem

(**The Maximum Degree Bound**) *For any graph G,* $\chi(G) \leq 1 + \Delta(G)$.

- This bound is not very good, since a single vertex with large degree can make it large.

Definition

A **deletion sequence** of a graph G is a sequence of its vertices formed by iterating the operation of deleting a vertex of smallest degree and adding it to the sequence until no vertices remain.

Degeneracy

Definition

A **construction sequence** of a graph is the reversal of a corresponding deletion sequence.
A graph is k-**degenerate** if its vertices can be successively deleted so that when deleted, each has degree at most k.
The **degeneracy** $D(G)$ of a graph G is the smallest k such that it is k-degenerate.

- A construction sequence may produce an efficient coloring.

Theorem

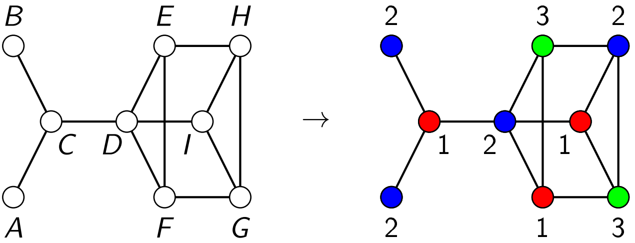
(**The Degeneracy Bound**) *For any graph G, $\chi(G) \leq 1 + D(G)$.*

Proof.

Greedily color a construction sequence of G. Each vertex has at most $D(G)$ neighbors when colored, so at most $1 + D(G)$ colors are needed. □

Example. A deletion sequence of the graph below left is the vertices A through I in alphabetical order.

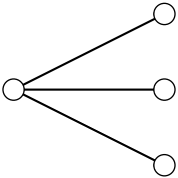
- Greedy coloring using the corresponding construction sequence produces the 3-coloring below right.
- This is a minimum coloring, one better than the 4-coloring guaranteed by the Degeneracy Bound.
- Note that beginning the construction sequence with HGI requires four colors.

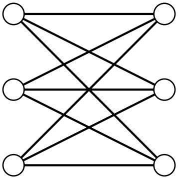


Comparing Bounds

- It is immediate from the definition of degeneracy that $\delta(G) \le D(G) \le \Delta(G)$.
- An immediate corollary to the Degeneracy Bound is $\chi(G) \le 1 + \Delta(G)$.
- The maximum degree bound is more famous, but is often much worse.

$K_{1,3}$





$K_{3,3}$

- For stars $K_{1,r}$, the Degeneracy Bound gives two, the correct value, while the maximum degree bound gives $r+$.
- A single vertex of large degree will determine the maximum degree bound, while only a subgraph with many large degree vertices determines the Degeneracy Bound.

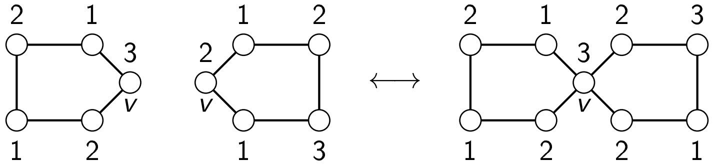
The Degeneracy Bound

- The Degeneracy Bound fails to give good results for some graphs, such as $K_{r,r}$.
- The Degeneracy Bound has been observed many times in various forms.
- The quantity $1 + D(G)$ has been called the **coloring number** (Erdos/Hajnal [1966]) and the **Szekeres-Wilf number** [1968].
- Unfortunately, the Degeneracy Bound is often presented in the confusing form $\chi(G) \leq 1 + \max\limits_{H \subseteq G} \delta(H)$.
- This seems to imply that all 2^n induced subgraphs of G must be checked.
- In fact, only one subgraph (the maximum core) must be checked, which can be done in $O(m)$ time.

- For a disconnected graph G with components G_i, they can be colored separately.
- Thus $\chi(G) = \max \chi(G_i)$.
- When a noncomplete graph has a cut-vertex, it decomposes into multiple blocks.
- The blocks can be colored separately, and the colorings can be permuted to agree on the cut-vertices.

Proposition

Let G be a graph with blocks B_i. Then $\chi(G) = \max \chi(B_i)$.



- We can characterize the extremal graphs for $D(G) \leq \Delta(G)$.

Proposition

Let G be a connected graph. Then $D(G) = \Delta(G)$ if and only if G is regular.

Proof.

($\Leftarrow$) If G is regular, then its maximum and minimum degrees are equal, so the result is obvious.
($\Rightarrow$) Let $D(G) = \Delta(G) = k$. Then G has a subgraph H with $\delta(H) = \Delta(G) \geq \Delta(H)$, so H is k-regular. If H were not all of G, then since G is connected, some vertex of H would have a neighbor not in H, implying that $\Delta(G) > \Delta(H) = \delta(H) = \Delta(G)$. But this is not the case, so $G = H$, and G is regular. □

- Among connected graphs, the Degeneracy Bound equals the Maximum Degree Bound only for regular graphs.

- The next theorem shows which regular graphs equal the Maximum Degree Bound.

Lemma

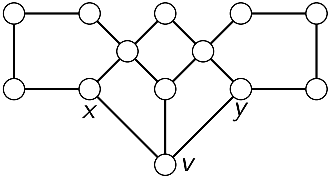
(Lovasz [1975]) *Given $r \geq 3$, if G is an r-regular 2-connected noncomplete graph, then G has a vertex v with two nonadjacent neighbors x and y such that $G - x - y$ is connected.*

Proof.

If G is 3-connected, let v be any vertex, and x and y be two nonadjacent neighbors of v, which must exist since G is noncomplete.
If $\kappa(G) = 2$, let $\{u, v\}$ be any 2-vertex-cut of G. Then $\kappa(G - v) = 1$, so $G - v$ has at least two end-blocks, and v has neighbors in all of them. Let x, y be two such neighbors. They must be nonadjacent, and $G - x - y$ is connected since blocks have no cut-vertices and $r \geq 3$. □

Proof.

If G is 3-connected, let v be any vertex, and x and y be two nonadjacent neighbors of v, which must exist since G is noncomplete.
If $\kappa(G) = 2$, let $\{u, v\}$ be any 2-vertex-cut of G. Then $\kappa(G - v) = 1$, so $G - v$ has at least two end-blocks, and v has neighbors in all of them. Let x, y be two such neighbors. They must be nonadjacent, and $G - x - y$ is connected since blocks have no cut-vertices and $r \geq 3$. □



Brooks' Theorem

Theorem

(**Brooks' Theorem** | Brooks [1941]) *If G is connected, then $\chi(G) = 1 + \Delta(G)$ if and only if G is complete or an odd cycle.*

Proof.

($\Leftarrow$) Equality certainly holds for cliques and odd cycles.
($\Rightarrow$) Let G satisfy the hypotheses. Then by Proposition 20, G is r-regular. The result certainly holds for $r \leq 2$, so we may assume $r \geq 3$. If G had a cut-vertex, each block could be colored with fewer than $r+1$ colors to agree on that vertex, so we may assume G is 2-connected and to the contrary, not a clique.
By the lemma, we can establish a deletion sequence for G starting with some vertex v and ending with its nonadjacent neighbors x and y so that all vertices but v have at most $r-1$ neighbors when deleted. Reversing this yields a construction sequence, and coloring greedily gives x and y the same color, using at most r colors. $\square$

- Thus the extremal graphs for $\chi(G) \leq 1 + \Delta(G)$ are complete graphs and odd cycles.

For the Degeneracy Bound, the extremal graphs include these, and also

- chordal graphs (trees, fans, max. outerplanar graphs)
- maximal k-degenerate graphs
- irregular graphs
- many more

No complete characterization of the extremal graphs for this bound is known.

Comparing Bounds

- There are many upper bounds on the chromatic number.
- When evaluating a bound, there are several questions to ask.
- What are the extremal graphs (those that make it an equality)?
- How efficiently can it be calculated?
- How does it compare to other bounds?
- If a bound is at least as good or better for every graph, we say it is superior.

The Small Clique Bound

Theorem

(Lovasz [1966]) Given t integers k_i such that $\sum k_i > \Delta(G)$, there is a vertex partition inducing subgraphs G_i so that $\Delta(G_i) < k_i$ for each i.

- When the clique number is small, split the graph into pieces, and color each of them using Brooks' Theorem.
- This produces the following bound.

Theorem

(Borodin/Kostochka [1977], Catlin [1978], Lawrence [1978]) Let G be a graph with $3 \leq \omega(G) \leq \Delta(G)$. Then $\chi(G) \leq \left\lceil \frac{\omega(G)}{\omega(G)+1}(\Delta(G)+1) \right\rceil$.

- The smallest values for which this improves on Brooks' Theorem are $\Delta = 7$ and $\omega \leq 3$.

The Independence Number Bound

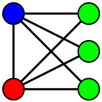
Proposition

For any graph G, $\chi(G) \leq n+1-\alpha(G)$.

Proof.

Color a vertex set with size $\alpha(G)$ using one color. Color each other vertex with a unique color. □

- A graph with $\chi(G) \leq n+1-\alpha(G)$ is contained in $K_{n-\alpha} + \alpha K_1$.



$K_2 + 3K_1$

- Since $D(K_{n-\alpha} + \alpha K_1) = n - \alpha$, $D(G) \leq n - \alpha(G)$.
- Thus the Degeneracy Bound is superior to the Independence Number Bound.

The Degree Sequence Bound

Proposition

(Welsh/Powell [1967]) Let G be a graph with degree sequence $d_1 \geq \ldots \geq d_n$. Then $\chi(G) \leq 1 + \max_i \min\{d_i, i-1\}$.

Proof.

Order the vertices by their degrees, and color them from largest to smallest. The i^{th} vertex colored has at most $\min\{d_i, i-1\}$ earlier neighbors. $\square$

- This bound is never superior to the Degeneracy Bound.

The Eigenvalue Bound

- Any graph can be represented as a matrix.
- The eigenvalues of a graph are well-defined.
- Let λ_1 be the largest eigenvalue of G.

Theorem

(Wilf [1967]) Let G be a connected graph. Then $\chi(G) \leq 1 + \lambda_1$, with equality exactly for complete graphs and odd cycles.

- The proof of this implicitly uses degeneracy.
- Szekeres and Wilf [1968] showed that $D(G) \leq \lambda_1(G)$, so the Degeneracy Bound is better.
- I showed [2010] when G is connected, $D(G) = \lambda_1(G)$ exactly for regular graphs.

The Longest Path Bound

- A tree is a graph that is connected and contains no cycle.

Theorem

If $\delta(G) \geq k$, then G contains all trees of size k.

- Let $l(G)$ be the length of the longest path of G.
- The previous theorem implies that $D(G) \leq l(G)$.

Proposition

For any graph G, $\chi(G) \leq 1 + l(G)$.

- Thus the Degeneracy Bound is superior to the Longest Path Bound.
- The extremal graph for the bound $D(G) \leq l(G)$ is K_n.

The Odd Cycle Bound

- Let l_o be the length of the longest odd cycle in a non-bipartite graph G.

Theorem

(Erdos/Hajnal [1966]) For any graph G containing an odd cycle, $\chi(G) \leq 1 + l_o$.

- Their proof used degeneracy implicitly.
- I have proved that if G is 2-connected, $D(G) \leq l_o$.
- This implies that $\chi(G) \leq 1 + D(G) \leq 1 + l_o$, so the Degeneracy Bound is superior to this bound.

Theorem

(Kenkre/Vishwanathan [2007]) For any graph G containing an odd cycle, $D(G) = l_o$ if and only if $G = K_{l_o+1}$, l_o odd.

Theorem

Let G be connected with order n, size m.
(Ershov/Kozhukhin [1962]) Then

$$\chi(G) \leq \left\lfloor \frac{3+\sqrt{9+8(m-n)}}{2} \right\rfloor$$

(Coffman et al [2003]) If $\delta(G) \geq 2$, and G is not a clique or an odd cycle, then

$$\chi(G) \leq \left\lfloor \frac{3+\sqrt{1+8(m-n)}}{2} \right\rfloor$$

(Me [2010]) If $\delta(G) \geq 3$, and $D(G) \leq n-3$, then

$$\chi(G) \leq 2+\sqrt{2m-3n+3}.$$

- The proofs of these results all use the Degeneracy Bound.

- There is another approach to graph coloring that is sometimes useful.

Algorithm

Find a maximum independent set S of a graph and color it with a single color. For $G - S$, repeat this step until all vertices are colored.

- This approach may not be optimal, as some graphs have no minimum coloring with any color class that is a maximum independent set.
- Finding a maximum independent set is not easy in general, so replacing 'maximum' with 'maximal' yields a faster algorithm.
- This algorithm does not translate directly into a bound, since most graphs have several maximum independent sets, and which is chosen may change the number of colors used.
- One bound based on this approach follows.

The Alpha-Omega Bound

Theorem

(Brigham/Dutton [1985]) *For any graph G,*
$\chi(G) \leq \frac{\omega(G)+n+1-\alpha(G)}{2}$.

Proof.

We use induction on n. Note the result is true for empty graphs, which have $\chi = n$, $\omega = 1$, and $\alpha = n$. Thus it holds for $n = 1$. Assume the result holds for graphs with fewer than n vertices, and let G be a nonempty graph with $n > 1$. Let S be a maximum independent set of G and $H = G - S$.

If H is complete, then
$\chi(G) = \omega(G) = \frac{\omega(G)+\omega(G)}{2} \leq \frac{\omega(G)+n+1-\alpha(G)}{2}$. If H is not complete, then $\alpha(H) \geq 2$, so

$$\chi(G) \leq \chi(H) + 1$$
$$\leq \frac{\omega(H)+n-\alpha(G)+1-\alpha(H)}{} + 1$$

Proof.

$$\begin{aligned}\chi(G) &\leq \chi(H)+1\\ &\leq \frac{\omega(H)+n-\alpha(G)+1-\alpha(H)}{2}+1\\ &\leq \frac{\omega(G)+n-\alpha(G)+1-2}{2}+1\\ &= \frac{\omega(G)+n-\alpha(G)+1}{2}.\end{aligned}$$

□

- This bound is superior to the Degeneracy Bound for some classes, such as $K_{r,s}$, and is inferior for many others.
- As noted before, it will not always be easy to calculate.

Reed's Conjecture

Theorem

(Schiermeyer [2007]) The extremal graphs for $\chi(G) \leq \frac{\omega(G)+n+1-\alpha(G)}{2}$ are those with $\alpha(G)+\omega(G) = n+1$, or $C_5 + K_{\omega-2} \subseteq G$ and $C_5 + K_{n-\omega-3} \subseteq \overline{G}$.

Proposition

(Reed [1998]) For any graph G, $\chi(G) \leq \frac{n+\omega(G)}{2}$.

- The bound is generally poor (it is always more than $\frac{n}{2}$).
- Reed conjectured a better bound.
- **Reed's Conjecture** [1998] is that $\chi(G) \leq \left\lceil \frac{\omega(G)+1+\Delta(G)}{2} \right\rceil$.
- If true, this would improve on the Degeneracy Bound for some graphs.

Other Applications

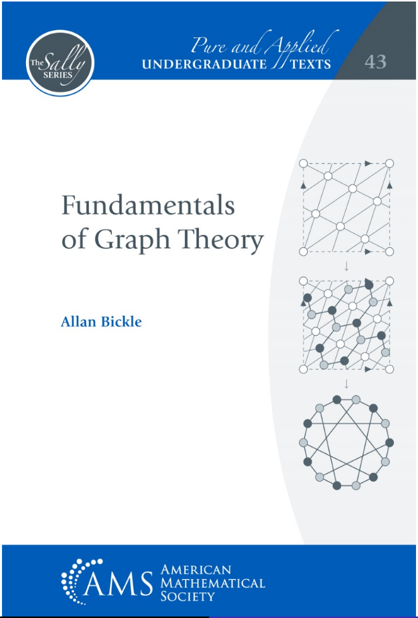
Degeneracy has applications to many other problems related to graph coloring.

- chromatic polynomials
- Nordhaus-Gaddum theorems

It can be used to prove good bounds on many other variations of graph coloring.

- list coloring
- L(2,1) Coloring
- vertex arboricity
- point partition number
- 2-tone chromatic number

Thank You!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