# Introduction to the k-Cores of a Graph

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Date

**Definition**

The k-core of a graph G is the maximal induced subgraph $H \subseteq G$ such that $\delta(H) \geq k$.

- The k-core was introduced by Steven B. Seidman in a 1983 paper entitled *Network structure and minimum degree*.

# Basics

**Proposition**

The $k$-core is well-defined.

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The cores are nested. That is, if $k > j$, then $C_k(G) \subseteq C_j(G)$.

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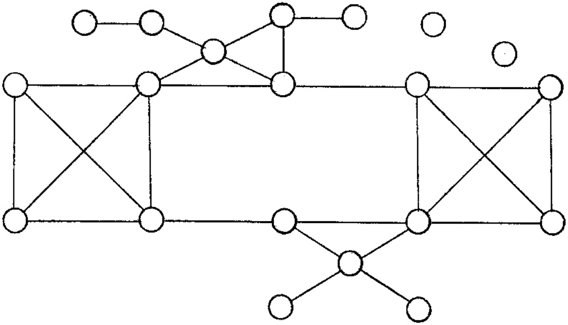
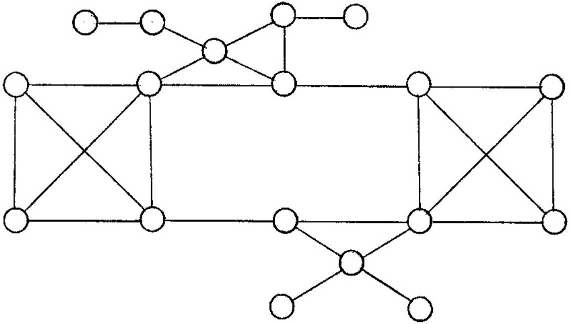
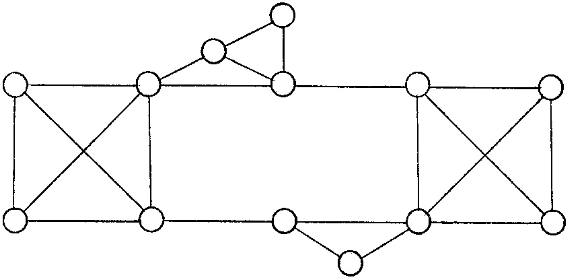


$G$ is its own 0-core.



The 1-core of $G$.

# Basics



The 2-core of $G$.

# Basics

The 3-core of $G$ is $2K_4$.

**Definition**

The core number of a vertex, $C(v)$, is the largest value for $k$ such that $v \in C_k(G)$.
The maximum core number of a graph, $\widehat{C}(G)$, is the maximum of the core numbers of the vertices of $G$.

- It is immediate that $\delta(G) \leq \widehat{C}(G) \leq \triangle(G)$.

**Proposition**

Let $G$ be a connected graph. Then $\widehat{C}(G) = \triangle(G) \iff G$ is regular.

**Proof.**

If $G$ is regular, then $\delta(G) = \triangle(G)$, so the result is obvious. For the converse, let $\widehat{C}(G) = \triangle(G) = k$. Then $G$ has a subgraph $H$ with $\delta(H) = \triangle(G) \geq \triangle(H)$, so H is $k$-regular. If $H$ were not all of $G$, then since $G$ is connected, some vertex of $H$ would have a neighbor not in $H$, implying that $\triangle(G) > \triangle(H) = \delta(H) = \triangle(G)$. But this is not the case, so $G = H$, and $G$ is regular. □

**Definition**

If the maximum core number and minimum degree of $G$ are equal, $\widehat{C}(G) = \delta(G)$, we say $G$ is $k$-monocore.

- We need a way to determine the $k$-core of a graph.

**The $k$-core algorithm (sketch)**

Input: graph $G$ with adjacency matrix $A$, integer $k$, degree array $D$.
Recursion: Delete all vertices with degree less than $k$ from $G$.
(That is, make a list of such vertices, zero out their degrees, and decrement the degrees of their neighbors.)
Result: The vertices that have not been deleted induce the $k$-core.

# Basics

**Theorem**

*Applying the $k$-core algorithm to graph $G$ yields the $k$-core of $G$, provided it exists.*

**Proof.**

Let $G$ be a graph and $H$ be the result of the algorithm.
Let $v \in H$. Then $v$ has at least $k$ neighbors in $H$. Then $\delta(H) \geq k$.
Then $H \subseteq C_k(G)$.
Let $v \in C_k(G)$. Then $v$ is an element of a set of vertices, each of which has at least $k$ neighbors in the set. None of these vertices will be deleted in the first iteration. If none have been deleted by the $n^{th}$ iteration, none will be deleted by the $n+1^{st}$ iteration. Thus none will ever be deleted. Thus $v \in H$. Thus $C_k(G) \subseteq H$.
Thus $H = C_k(G)$, so the algorithm yields the $k$-core. □

**Theorem**

*[Batagelj/Zaversnik 2003] The k-core algorithm has efficiency $O(m)$. (That is, it is linear on the size m.)*

- This depends on using an edge list as the data structure.

**Definition**

A vertex deletion sequence of a graph $G$ is a sequence that contains each of its vertices exactly once and is formed by successively deleting a vertex of smallest degree.

- We may wish to construct a graph by successively adding vertices of relatively small degree.

**Definition**

A vertex construction sequence of a graph is the reversal of a deletion sequence.

**Definition**

A graph is $k$-degenerate if its vertices can be successively deleted so that when deleted, each has degree at most $k$. The degeneracy of a graph is the smallest $k$ such that it is $k$-degenerate.

- Thus the $k$-core algorithm implies a natural min-max relationship.

**Corollary**

*For any graph, its maximum core number is equal to its degeneracy.*

**Proof.**

Let $G$ be a graph with degeneracy d and $k = \widehat{C}(G)$. Since $G$ has a $k$-core, it is not $k-1$-degenerate, so $k \leq d$. Since $G$ has no $k+1$-core, running the $k$-core algorithm for the value $k+1$ destroys the graph, so $G$ is $k$-degenerate, and $k = d$. □

# Basics

- Many important classes of graphs are monocore.

| Class of Graphs | Maximum Core Number |
|---|---|
| $r$-regular | $r$ |
| nontrivial trees | 1 |
| forests (no trivial components) | 1 |
| complete bipartite $K_{a,b}$, $a \leq b$ | $a$ |
| $K_{a_1,\ldots,a_n}$, $a_1 \leq a_2 \leq \ldots \leq a_n$ | $a_1 + \ldots + a_{n-1}$ |
| wheels | 3 |
| maximal outerplanar, $n \geq 3$ | 2 |

- For more general classes of graphs, we may only be able to bound the maximum core number.

**Proposition**

If G is planar, $\widehat{C}(G) \leq 5$. If $G$ also has order $n < 12$, then $\widehat{C}(G) \leq 4$.

**Proof.**

If there were a planar 6-core, it would have $2m = \sum d(v_i) \geq 6n$, that is, $m \geq 3n$. But every planar graph has $m \leq 3n - 6$.
Let planar graph $G$ have a 5-core $H$, where $H$ has order n, size m. Then $2m = \sum d(v_i) \geq 5n$, so $m \geq \frac{5}{2}n$. Since $H$ is planar, $m \leq 3n - 6$. Thus $\frac{5}{2}n \leq 3n - 6$, so $n(G) \geq n(H) \geq 12$. □

# k-Shells

## Definition

For $k > 0$, the $k$-shell of a graph $G$, $S_k(G)$, is the subgraph of $G$ induced by the edges contained in the $k$-core and not contained in the $k+1$-core. For $k = 0$, the 0-shell of $G$ is the vertices of the 0-core not contained in the 1-core.

## Definition

The $k$-boundary of G, $B_k(G)$, is the set of vertices contained in both the $k$-shell and the $k+1$-core.

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**Definition**

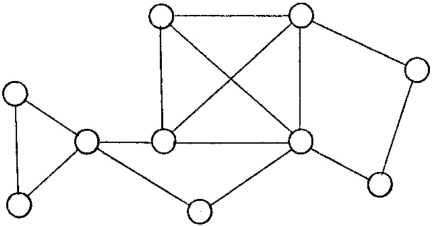
The $k$-boundary of G, $B_k(G)$, is the set of vertices contained in both the $k$-shell and the $k+1$-core.

## Definition

The proper $k$-shell of $G$, $S'_k(G)$, is the subgraph of $G$ induced by the non-boundary vertices of the $k$-shell. The order of the $k$-shell of $G$ is defined to be the order of the proper $k$-shell.

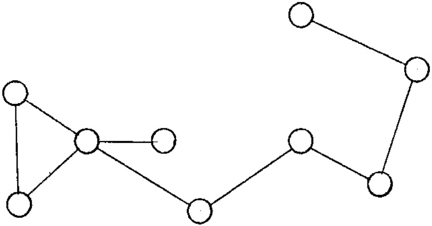
- Thus the vertices of the proper $k$-shells partition the vertex set of $G$. A vertex has core number $k$ if and only if it is contained in the proper $k$-shell of $G$. The proper $k$-shell is induced by the vertices with core number $k$.

# k-Shells

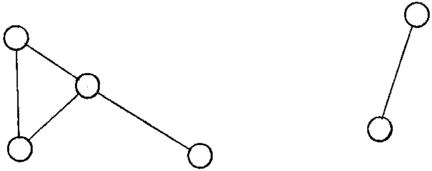


A graph $G$.

# k-Shells



The 2-shell of $G$.



The proper 2-shell of $G$.

# k-Shells

## Theorem

*A graph $F$ with vertex subset $B$ can be a $k$-shell of a graph with boundary set $B$ if and only if no component of $F$ has vertices entirely in $B$, $\delta(B) \geq 1$, $\delta_F(V(F) - B) = k$, and $F$ contains no subgraph $H$ with $\delta_H(V(H) - B) \geq k + 1$.*

## Proof.

($\Rightarrow$) Let $F$ be a $k$-shell of graph $G$ with boundary set $B$. If any component of $F$ had all vertices in $B$, it would be contained in the $k+1$-core of $G$. $F$ is induced by edges, so $\delta(B) \geq 1$. If a vertex $v$ in $F$ and not in $B$ had $d(v) < k$, it would not be in the $k$-core of $G$. If $F$ had such a subgraph $H$, it would be contained in the $k+1$-core of $G$.

($\Leftarrow$) Let $F$ be a graph satisfying these conditions. Overlap each vertex in $B$ with a distinct vertex of a $k+1$-core $G$ with sufficiently large order. Then $F$ is the $k$-shell of the resulting graph. □

**Corollary**

*The 1-shell of $G$, if it exists, is a forest with no trivial components and at most one boundary vertex per component.*

**Proof.**

$F$ is acyclic, $\delta(F) = 1$, and two boundary vertices in a tree are connected by a path, which would be in the 2-core.

□

**Theorem**

*A graph $F$ can be a proper $k$-shell if and only if $F$ does not contain a $k+1$-core.*

**Proof.**

The forward direction is obvious.
Let $F$ be a graph that does not contain a $k+1$-core. Let $M$ be a $k+1$-core. For each vertex $v$ in $F$, let $a(v) = \max\{k - d(v), 0\}$. For each vertex $v$, take $a(v)$ copies of $M$ and link each to $v$ by an edge between $v$ and a vertex in $M$. The resulting graph $G$ has $F$ as its proper $k$-shell.

□

**Corollary**

*A graph $F$ can be a proper 1-shell if and only if $F$ is a forest.*

# k-Shells

**Proposition**

The size m of a $k$-shell with order n satisfies $\left\lceil \frac{k \cdot n}{2} \right\rceil \leq m \leq k \cdot n$.

**Proof.**

The non-boundary vertices of the $k$-shell of $G$ can be successively deleted so that when deleted, they have degree at most $k$. Thus $m \leq k \cdot n$. The non-boundary vertices have degree at least $k$, so there are at least $\frac{k \cdot n}{2}$ edges.

□

**Corollary**

*Let $s_k$ be the order of the $k$-shell of $G$, $0 \leq k \leq r = \widehat{C}(G)$. Then the size $m$ of $G$ satisfies*

$$\sum_{k=1}^{r} \left\lceil \frac{k \cdot s_k}{2} \right\rceil \leq m \leq \sum_{k=1}^{r} k \cdot s_k - \binom{k+1}{2}.$$

# k-Shells

## Proposition

The size m of a $k$-shell with order n and b boundary vertices satisfies

$$\left\lceil \frac{k \cdot n + b}{2} \right\rceil \leq m \leq k \cdot n - \binom{k-b+1}{2}.$$

## Proof.

When deleted, the $i^{th}$ to last vertex can have degree at most $b + i - 1$. Thus the upper bound must be reduced by $\sum_{i=1}^{k-b} i = \frac{(k-b)(k-b+1)}{2} = \binom{k-b+1}{2}$. The boundary vertices each contribute degree at least one to the lower bound. The result follows. □

**Corollary**

*Let $s_k$ be the order of the k-shell of $G$ and $b_k$ be the order of the k-boundary of $G$, $0 \leq k \leq r = \widehat{C}(G)$. Then the size $m$ of $G$ satisfies*

$$\sum_{k=1}^{r} \left\lceil \frac{k \cdot s_k + b_k}{2} \right\rceil \leq m \leq \sum_{k=1}^{r} \left( k \cdot s_k - \binom{k - b_k + 1}{2} \right).$$

## Proposition

If $G$ is connected, then its 2-core is connected.

## Proof.

Let $G$ be connected, and $u, v \in C_2(G)$. Then there is a $u - v$ path in $G$. The vertices on the path all have degree at least two, and all are adjacent to at least two vertices in a set with minimum degree two, since u and v are in the 2-core of $G$. Thus the $u - v$ path is in the 2-core of $G$, so it is connected.

□

**Theorem**

*A vertex $v$ of $G$ is contained in the 2-core of $G$ if and only if $v$ is on a cycle or $v$ is on a path between vertices of distinct cycles.*

**Proof.**

($\Leftarrow$) Let $v$ be on a cycle or a path between vertices of distinct cycles. Both such graphs are themselves 2-cores, so v is in the 2-core of $G$.

($\Rightarrow$) Let $v$ be in the 2-core of $G$. If $v$ is on a cycle, we are done. If not, then consider a longest path $P$ in the 2-core through $v$. All the edges incident with v must be bridges, so $v$ is in the interior of $P$. An end-vertex $u$ of $P$ must have another neighbor, which cannot be a new vertex, so it must be on $P$. If its neighbor were on the opposite side of $v$, then $v$ would be on a cycle. Thus its neighbor must be between $u$ and $v$ on $P$. Repeating this argument for the other end of $P$ shows that $v$ is on a path between vertices on cycles. □

### Corollary

*A graph $G$ is a 2-core $\iff$ every end-block of $G$ is 2-connected.*

### Proof.

If every end-block of $G$ is 2-connected, then every vertex of $G$ is either on a cycle or a path between cycles. Thus $G$ is a 2-core. If some end-block of $G$ is not 2-connected, then it is $K_2$, so $G$ has a vertex of degree one and is not a 2-core. □

### Definition

A block-tree decomposition of a 2-core $G$ is a decomposition of $G$ into 2-connected blocks and trees so that if $T$ is nontrivial, each end-vertex of $T$ is shared with a distinct 2-connected block, if $T$ is trivial, it is a cut-vertex of at least two 2-connected blocks, and there are no two disjoint paths between two distinct blocks.

# The Structure of k-Cores

**Corollary**

*Every 2-core has a unique block-tree decomposition.*

**Proof.**

Let $F$ be the subgraph of a 2-core $G$ induced by the bridges and cut-vertices of $G$. Then $F$ is acyclic, so it is a forest. Break each component of $F$ into branches at any vertex contained in a component of $G - F$. Also break $G - F$ into blocks, which must be 2-connected. By the previous corollary, each end-vertex of each of the trees must overlap a 2-connected block. If any block contained two end-vertices of the same tree, then there would be a cycle containing edges from the tree. If there were two disjoint paths between two blocks, they wouldn't be distinct. This decomposition is unique because the block decomposition of a graph is unique and any blocks that are $K_2$ and on a path between 2-connected blocks that does not go through any other 2-connected blocks must be in the same tree.

**Theorem**

*[Whitney, see West p. 163] A graph is 2-connected $\Longleftrightarrow$ it has an ear decomposition. Every cycle is the cycle in some ear decomposition.*

**Theorem**

*[Bollobas p. 15] Let $G$ be a minimally 2-connected graph that is not a cycle. Let $D \subset V(G)$ be the set of vertices of degree two. Then $F = G - D$ is a forest with at least two components. Each component $P$ of $G[D]$ is a path and the end-vertices of $P$ are not joined to the same tree of the forest $F$.*

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# The Structure of k-Cores

## Corollary

*A graph $G$ which is not a cycle is minimally 2-connected $\iff$ it has an ear decomposition with each path of length at least 2, no ear joined to vertices in a single component of $F$, and no ear connects or creates a cycle in $F$.*

## Proof.

($\Rightarrow$) Let $G$ be minimally 2-connected. Then $G$ has an ear decomposition. A path of length one in the ear decomposition would be an essential edge. So would an edge between vertices in a component of $F$ that are the ends of an ear. The final condition is implied by the second theorem.

($\Leftarrow$) Assume the hypothesis. The ear decomposition implies that $G$ is 2-connected. Adding the first ear makes $F$ disconnected, and adding subsequent ears keep it a forest. The ears must connect different components of $F$. By the previous theorem, $G$ is minimal. □

**Theorem**

*A graph $G$ is a connected 2-core $\iff$ it is contained in the set $S$ whose members can be constructed by the following rules.*
*1. All cycles are in $S$.*
*2. Given one or two graphs in $S$, the result of joining the ends of a (possibly trivial) path to it or them is in $S$.*

**Proof.**

($\Leftarrow$) A cycle has minimum degree 2, and applying step 2 does not create any vertices of lower degree, so a graph in $S$ is a 2-core.
($\Rightarrow$) This is clearly true if $G$ has order 3. Assume the result holds for orders up to r, and let $G$ have order $r+1$. Let $P$ be an ear or cut-vertex of $G$. Making $P = K_2$ is only necessary when $G$ has minimum degree at least 3 and is 2-connected. In this case, edges can be deleted until one of these conditions fails to hold. Then if $P$ has internal vertices, deleting them results in a component or components with order at most r. The same is true if $P$ is a cut-vertex, and $G$ is split into blocks. Then the result follows by induction. □

**Theorem**

*The set of connected 2-monocore graphs is equivalent to the set $S$ of graphs that can be constructed using the following rules.*
*1. All cycles are in $S$.*
*2. Given one or two graphs in $S$, the graph $H$ formed by identifying the ends of a path of length at least two with vertices of the graph or graphs is in $S$.*
*3. Given a graph $G$ in $S$, form $H$ by taking a cycle and either identifying a vertex of the cycle with a vertex of $G$ or adding an edge between one vertex in each.*

**Proof.**

(⇐) We first show that if $G$ is in $S$, then $G$ is 2-monocore. Certainly cycles are 2-monocore. Let $H$ be formed from $G$ in $S$ by applying rule 2. Then $H$ has minimum degree 2 and since $G$ is 3-core-free and internal vertices of the path have degree 2, $H$ is also 3-core-free. Thus $H$ is 2-monocore. The same argument works for adding a path between two graphs. Let $H$ be formed from $G$ in $S$ by applying rule 3. Then $H$ has minimum degree 2 and since $G$ is 3-core-free and all but one vertex of the cycle have degree 2, $H$ is also 3-core-free. Thus $H$ is 2-monocore.

(⇒) We now show that if $G$ is 2-monocore, it is in $S$. This clearly holds for all cycles, including $C_3$, so assume it holds for all 2-monocore graphs of order up to r. Let $G$ be 2-monocore of order $r+1$ and not a cycle. Then $G$ has minimum degree 2, so it has a vertex $v$ of degree 2. Then $v$ is contained in $P$, an ear of length at least 2, or $C$, a cycle which has all but one vertex of degree 2.

**Proof.**

Case 1. $G$ has an ear $P$. If $G - P$ is disconnected, then the components of $G$ are 2-monocore, and hence in $S$. Then $G$ can be formed from them using rule 2, so $G$ is in $S$. If $G - P$ is connected, then it is 2-monocore, and hence in $S$. Then $G$ can be formed from $G - P$ using rule 2, so $G$ is in $S$.

Case 2. We may assume that $G$ has no such ear $P$. Then $G$ has a cycle $C$ with all but one vertex of degree 2, and one vertex $u$ of degree more than 2. If $u$ has degree at least 4 in $G$, then let $H$ be formed by deleting all the vertices of $C$ except $u$. Then H is 2-monocore, and $G$ can be formed from it using rule 3. If $d(u) = 3$, then its neighbor not in the cycle has degree at least three, so $G - C$ is 2-monocore, and $G$ can be formed from it by using rule 3. □

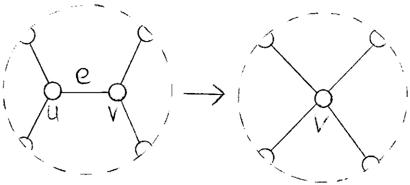
**Corollary**

*The set of 2-shells is equivalent to the set $S'$ of graphs constructed using the following rules.*
*1. All graphs in set $S$ from the previous theorem and all 3-cores are in $S'$.*
*2. Given one or two graphs in $S'$, the graph $H$ formed by identifying the ends of a path of length at least two with vertices of the graph or graphs is in $S'$.*
*3. Given a graph $G$ in $S'$, form $H$ by taking a cycle and either identifying a vertex of the cycle with a vertex of $G$ or adding an edge between one vertex in each.*
*Finally, delete the 3-cores (keeping boundary vertices) last.*

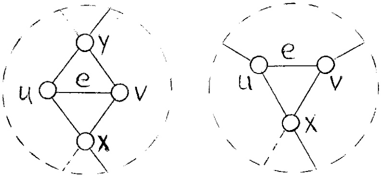
# The Structure of k-Cores

## Theorem

*Every 3-core has $K_4$ as a minor.*




Cases 1, 2, and 3.



# The Structure of k-Cores

**Theorem**

*Let $H$ be a graph with $\triangle(H) \leq 3$. Then $G$ has $H$ as a minor $\Longleftrightarrow$ $G$ has a subdivision of $H$.*

**Corollary**

*Every 3-core contains a subdivision of $K_4$.*

**Corollary**

*Every end-block of a 3-core contains a subdivision of $K_4$.*

**Proof.**

A subdivision of $K_4$ cannot contain a cut-vertex, so it must be contained in some block of a 3-core. Form a graph with two copies of an end-block of a 3-core by identifying their unique cut-vertices. The graph that results is a 3-core, so it has a subdivision of $K_4$ in a block.

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# The Structure of k-Cores

**Proposition**

[Chartrand/Lesniak p. 72] Let $G$ be a $k$-core of order n and $1 \leq l \leq n-1$. If $k \geq \left\lceil \frac{n+l-2}{2} \right\rceil$, then $G$ is $l$-connected.

**Proof.**

Assume the hypothesis. Then the sum of the degrees of any two nonadjacent vertices of $G$ is at least $n+l-2$, so they have at least $l$ common neighbors. Thus $G$ is $l$-connected. □

**Corollary**

*Let $G$ be a $k$-core with order $n$. If $k+1 < n < 2k+2$, then $diam(H) = 2$.*

**Proof.**

Assume the hypothesis. Since $k < n-1$, $G$ is not complete, so its diameter is at least 2. By the previous result, any pair of nonadjacent vertices has a common neighbor since $n \leq 2k - 1 + 2$. Thus $diam(G) = 2$.

□

### Theorem

*[Chartrand/Lesniak page 294] Let $n \geq r \geq 2$. Then every graph of order $n$ and size at least $\left\lfloor \left(\frac{r-2}{2r-2}\right) n^2 \right\rfloor + 1$ contains $K_r$ as a subgraph.*

### Corollary

*[Seidman 1983] A k-core with order $n$ must contain a clique $K_r$ as a subgraph if $n < \left(\frac{r-1}{r-2}\right) k$.*

### Proof.

Let H be a $k$-core with order $n < \left(\frac{r-1}{r-2}\right) k$. Then $k > \left(\frac{r-2}{r-1}\right) n$, so $H$ has size m with

$$m \geq \frac{n \cdot k}{2} > \left(\frac{r-2}{2r-2}\right) n^2 \geq \left\lfloor \left(\frac{r-2}{2r-2}\right) n^2 \right\rfloor .$$

Thus $m \geq \left\lfloor \left(\frac{r-2}{2r-2}\right) n^2 \right\rfloor + 1$, so by the previous theorem, $H$ contains $K_r$ as a subgraph.

**Theorem**

*[Seidman 1983] Let H be a connected k-core with order* $n \geq 2k+2$ *and connectivity* $l$*, then*

$$diam(H) \leq 3\left\lfloor \frac{p-2k-2}{\beta} \right\rfloor + b(n,k,l) + 3$$

*where* $\beta = \max\{k+1, 3l\}$ *and* $r$ *is the element of* $\{0,\ldots,\beta-1\}$ *such that* $r \equiv n-2k-2 \pmod{\beta}$ *and*

$$b(n,k,l) = \begin{cases} 0 & 0 \leq r < l \\ 1 & l \leq r < 2l \\ 2 & 2l \leq r \end{cases}.$$

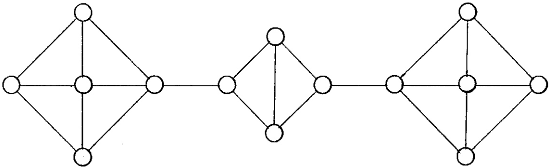
**Corollary**

*[Moon 1965] If H has order $n \geq 2k+2$, then*

$$diam(H) \leq 3\left\lfloor \frac{n}{k+1} \right\rfloor + a(p,k) - 3,$$

*where*

$$a(p,k) = \begin{cases} 0 & p \equiv 0 \pmod{k+1} \\ 1 & p \equiv 1 \pmod{k+1} \\ 2 & else \end{cases}.$$



# Thank You

Thank You!