A New Spin on Cyclic Decompositions

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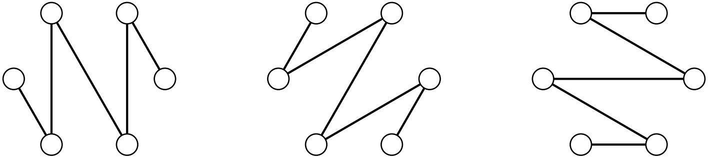
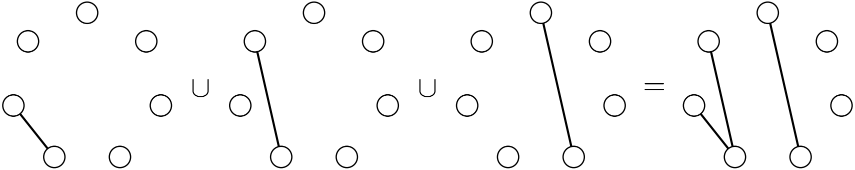
Cyclic Decompositions

- Most textbooks introduce cyclic decompositions informally.

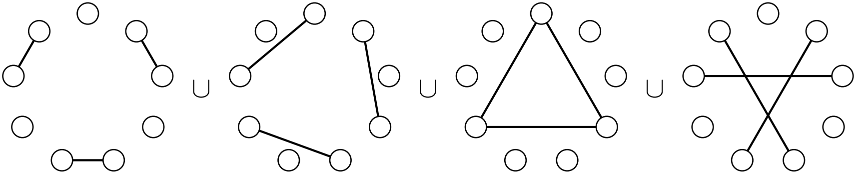


- We examine how to construct cyclic decompositions.
- First consider an odd vertex cycle.
- Number the vertices of K_{2r+1} from 0 to $2r$.

- Let the **length of an edge** ij be whichever of $i - j$ and $j - i$ has the smallest positive residue mod n.
- The possible lengths range from 1 to r.
- Each set of edges with the same length is an orbit containing $2r + 1$ edges.
- One way to construct a cyclic decomposition is to pick one edge from each orbit, and let G be the graph induced by this edge set.
- Repeatedly increasing the indices by 1 (mod n) produces $2r + 1$ copies of G that cyclically decompose K_{2r+1}.



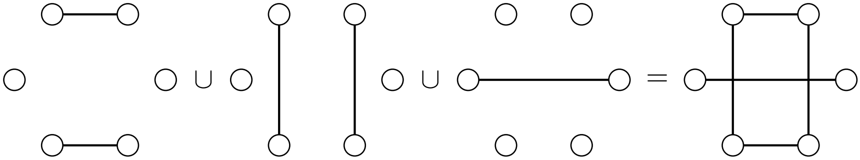
Cyclic Decompositions



- More generally, if $2r+1 = ks$, we could pick exactly s edges from each orbit that are k steps apart in each orbit, producing a cyclic k-decomposition.
- A factor could use no edges from an orbit, but it must have the same number of edges from each orbit if it has any.

Cyclic Decompositions

- Now consider an even vertex cycle, numbered from 0 to $2r-1$.
- There are $r-1$ sets of $2r$ edges with lengths 1 to $r-1$.
- However, the set of edges with length r (the **diameters**) has only r edges.
- Thus if we choose one edge from the diameters, we must choose two edges r steps apart from each other orbit.
- In general, if we choose s edges from the diameters, we must choose $2s$ edges from each other orbit from which we select edges.



- Constructing cyclic decompositions isn't difficult, but the opposite problem of determining whether a given graph cyclically decomposes a given complete graph is more challenging.
- If a graph can use exactly one edge from each class, it can cyclically decompose K_{2m+1}.
- Vertex labelings can be used to characterize these cyclic decompositions.

Definition

A labeling $f(v) : V(G) \to \{0, ..., 2m\}$ of a graph G with size m is a ρ-**labeling** if exactly one of each pair $\{i, 2m+1-i\}$, $1 \leq i \leq m$, occurs as a difference $|f(u) - f(v)|$ for $uv \in E(G)$.
A labeling $f(v) : V(G) \to \{0, ..., m\}$ of the vertices of a graph G is a **graceful labeling** if each value in $\{1, ..., m\}$ occurs exactly once as a difference $|f(u) - f(v)|$ for $uv \in E(G)$. A graph is **graceful** if it has a graceful labeling.

Graph Labeling

- Much work on these labelings has focused on trees.
- Ringel [1964] conjectured that any tree T with size m decomposes K_{2m+1}.
- Kotzig conjectured that there is always a cyclic T-decomposition of K_{2m+1}.
- Alexander Rosa stated that this is equivalent to the conjecture that every tree has a ρ-labeling.
- Rosa also published the following stronger conjecture.

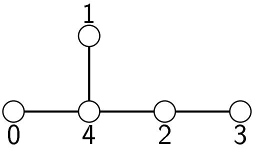
Conjecture

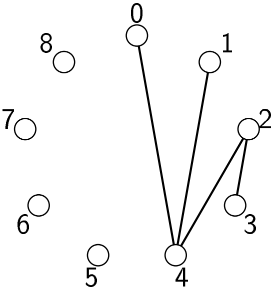
(**Graceful Tree Conjecture** | Rosa [1967]) *Every tree is graceful.*

- Montgomery, Pokrovskiy, and Sudakov [2020] proved Ringel's Conjecture and Kotzig's Conjecture for sufficiently large n.

Rosa's Characterization

- Rosa used "β-labeling" for what would later be called graceful labeling.
- Clearly every graceful labeling is also a ρ-labeling. A graceful labeling of a tree with size four and the corresponding cyclic decomposition of K_9 are illustrated below.



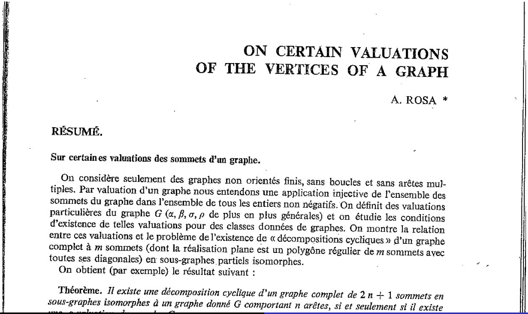


Rosa's Characterization

Theorem

A cyclic decomposition of K_{2m+1} into copies of a graph G with size m exists if and only if G has a ρ-labeling.

- This theorem was stated by Rosa [1967].
- I wanted to see a proof, but none of the secondary sources I consulted had one.
- Eventually, I got hold of Rosa's original paper.



ON CERTAIN VALUATIONS
OF THE VERTICES OF A GRAPH

A. ROSA *

RÉSUMÉ.

Sur certaines valuations des sommets d'un graphe.

On considère seulement des graphes non orientés finis, sans boucles et sans arêtes multiples. Par valuation d'un graphe nous entendons une application injective de l'ensemble des sommets du graphe dans l'ensemble de tous les entiers non négatifs. On définit des valuations particulières du graphe G ($\alpha, \beta, \sigma, \rho$ de plus en plus générales) et on étudie les conditions d'existence de telles valuations pour des classes données de graphes. On montre la relation entre ces valuations et le problème de l'existence de « décompositions cycliques » d'un graphe complet à m sommets (dont la réalisation plane est un polygône régulier de m sommets avec toutes ses diagonales) en sous-graphes partiels isomorphes.
On obtient (par exemple) le résultat suivant :

Théorème. *Il existe une décomposition cyclique d'un graphe complet de $2n+1$ sommets en sous-graphes isomorphes à un graphe donné G comportant n arêtes, si et seulement si il existe*

Rosa's Definitions

- Let's examine the theorem there.
- First, we need the definition of cyclic decomposition.

Now let us show the connection between the introduced valuations and the so-called cyclic decompositions of the complete graph into isomorphic subgraphs.

By a length of an edge (v_i, v_j) in the graph $\langle n \rangle$ $(i, j = 1, 2, \ldots, n; i \neq j)$ we mean a number $d_{ij} = \min(|i - j|, n - |i - j|)$. By a turning of an edge (v_i, v_j) in a graph $\langle n \rangle$ we understand the increase of both indices by one, so that from the edge (v_i, v_j) we obtain the edge (v_{i+1}, v_{j+1}), the indices taken modulo n. By a turning of a subgraph G in a graph $\langle n \rangle$ we understand the simultaneous turning of all edges of G.

By a decomposition of the complete graph $\langle n \rangle$ we mean an edge-disjoint decomposition, i. e., a system R of subgraphs such that any edge of the graph $\langle n \rangle$ belongs to exactly one of the subgraphs of R. A decomposition R of a graph $\langle n \rangle$ is said to be cyclic, if the following holds : If R contains a graph G, then it contains also the graph G' obtained by turning G.

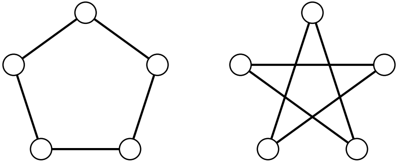
Theorem 7. *A cyclic decomposition of the complete graph $\langle 2n + 1 \rangle$ into subgraphs isomorphic to a given graph G with n edges exists if and only if there exists a ρ-valuation of the graph G.*

Rosa's Definitions

- According to Rosa:
- "By turning of an edge (v_i, v_j) in a graph $\langle n \rangle$ we understand the increase of both indices by one, so that from the edge (v_i, v_j) we obtain the edge (v_{i+1}, v_{j+1}). By turning of a subgraph G in a graph $\langle n \rangle$ we understand the simultaneous turning of all edges of G."
- Note that Rosa is only allowing a single vertex cycle in this definition.
- "A decomposition R of a graph $\langle n \rangle$ is said to be cyclic, if the following holds: If R contains a graph G, then it contains also the graph G' obtained by turning G."
- There is some ambiguity in this definition!
- Can $G = G'$, or must they be distinct?

Rosa's Definitions

- Is this decomposition cyclic?



- Definition A: Cyclic vertex permutation, factors map to factors
- Definition B: Cyclic vertex permutation, factors map to factors (distinct unless applied $2n+1$ times)
- Note that in both definitions, the permutation of factors need not be cyclic.

Rosa's Proof

- Which definition was intended?
- Let's examine Rosa's proof and see if context will answer the question.

I. The sufficiency is almost evident. Let a_i be the value of the vertex v_i in a ρ-valuation O_G of the graph G with n edges. Let us denote the vertices of the complete graph $\langle 2n+1 \rangle$, so that $v_i = v_{a_i}$. Then

$$d_{ij} = \begin{cases} b_k & \text{if} \quad b_k \leqslant n \\ 2n+1-b_k & \text{if} \quad b_k > n \end{cases}$$

where b_k is the value of the edge h_k of G in O_G and d_{ij} is the length of the edge h_k in the graph $\langle 2n+1 \rangle$. This implies that the edges of G have in the graph $\langle 2n+1 \rangle$ mutually different lengths, which again implies the existence of a cyclic decomposition of the complete graph $\langle 2n+1 \rangle$ into subgraphs isomorphic to G, the last obtained by turning consecutively the graph G $2n$ times in $\langle 2n+1 \rangle$.

- Showing that a ρ-labeling produces a cyclic decomposition of K_{2n+1} is routine.

II. Let a cyclic decomposition of the complete graph $\langle 2n+1 \rangle$ into subgraphs isomorphic to G be given. Let us take an arbitrary subgraph G_+ (G_+ is isomorphic to G) of $2n+1$ subgraphs of this decomposition and prove that the edges of G_+ have mutually different lengths in the graph $\langle 2n+1 \rangle$. Suppose that G_+ contains two edges of equal length i, $1 \leqslant i \leqslant n$, for example (v_x, v_{x+i}), (v_y, v_{y+i}), $x \neq y$ (without loss on generality we can assume $y > x$). By the definition of a cyclic decomposition, this decomposition must also contain a graph $G_+^{(y-x)}$ obtained from G by turning it $y-x$ times. The graph $G_+^{(y-x)}$ contains the edge (v_y, v_{y+i}), which is a contradiction to the definition of a decomposition of a graph. So all the edges of G_+ have mutually different lengths in the graph $\langle 2n+1 \rangle$, which means that there exists a ρ-valuation of G.

- Rosa considers decomposing K_{2n+1} into $2n+1$ factors of size n.
- He wants to show that each edge has a distinct length.
- He assumes to the contrary that a factor contains two edges of equal length.

Rosa's Proof

> II. Let a cyclic decomposition of the complete graph $\langle 2n+1 \rangle$ into subgraphs isomorphic to G be given. Let us take an arbitrary subgraph G_+ (G_+ is isomorphic to G) of $2n+1$ subgraphs of this decomposition and prove that the edges of G_+ have mutually different lengths in the graph $\langle 2n+1 \rangle$. Suppose that G_+ contains two edges of equal length i, $1 \leqslant i \leqslant n$, for example (v_x, v_{x+i}), (v_y, v_{y+i}), $x \neq y$ (without loss on generality we can assume $y > x$). By the definition of a cyclic decomposition, this decomposition must also contain a graph $G_+^{(y-x)}$ obtained from G by turning it $y - x$ times. The graph $G_+^{(y-x)}$ contains the edge (v_y, v_{y+i}), which is a contradiction to the definition of a decomposition of a graph. So all the edges of G_+ have mutually different lengths in the graph $\langle 2n+1 \rangle$, which means that there exists a ρ-valuation of G.

- He then points out that one of the edges would appear in a (different) copy of G formed by turning G.
- He concludes that this is a contradiction.
- This proof only works if edges of a given length can only be used once (Definition B).

A More General Proof

- I have a proof that works for Definition A.

Theorem

A cyclic decomposition of K_{2m+1} into copies of a graph G with size m exists if and only if G has a ρ-labeling.

Proof.

($\Rightarrow$) If a cyclic decomposition of K_{2m+1} into copies of a graph G with size m exists, then it contains $2m+1$ factors. The edges with length l form an orbit, so if G contains r edges of length l, then r divides $2m+1$. Further, if G contains an edge of length l', then it must contain r such edges. Thus r divides m. But $\gcd(m, 2m+1) = 1$, so $r = 1$. Thus G contains at most one edge of each length, hence exactly one. Thus G has a ρ-labeling. □

Other Authors' Definitions

- Which definition of cyclic decomposition do later authors use?
- Fu/Wu [2004] have the following definition.
- "An automorphism of a STS (V, B) is a bijection such that $\{x, y, z\} \in B$ if and only if $\{\alpha(x), \alpha(y), \alpha(z)\} \in B$. A STS(v) is cyclic if it has an automorphism that is a permutation consisting of a single cycle of length v, for example $(1, 2, 3, \dots v)$."
- This seems consistent with Rosa (Definition A or B).

Other Authors' Definitions

- A survey article by Saad El-Zanati and Charles Vanden Eynden [2009] contains the following definitions.
- "by clicking we mean applying the isomorphism $(i,j) \to (i+1, j+1)$."
- "Such a G-decomposition Δ is cyclic (purely cyclic) if clicking is a permutation (t-cycle) of Δ."
- The first definition is the same as Rosa (Definition A or B).
- Definition C (purely cyclic): Cyclic vertex permutation, induces cyclic permutation of factors.
- They state Rosa's result as:

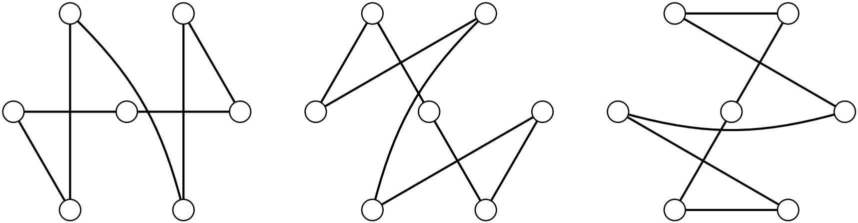
Theorem

Let G be a graph with m edges. There exists a purely cyclic decomposition of K_{2m+1} if and only if G has a ρ-labeling.

Other Authors' Definitions

- Perhaps the most definitive source on graph labeling is Joseph Gallian's survey article [2019].
- "Rosa [2136] proved that a cyclic decomposition of the edge set of the complete graph K_{2q+1} into subgraphs isomorphic to a given graph G with q edges exists if and only if G has a ρ-labeling. (A decomposition of K_n into copies of G is called cyclic if the automorphism group of the decomposition itself contains the cyclic group of order n.)"
- Definition D: Vertex permutation, induces cyclic permutation of factors (order n).
- Order n implies a single vertex cycle, so this is Definition C when there must be n factors.

- Note that according to Definitions A-D, the **Walecki decomposition** below is not cyclic.



- There is a definition that accounts for this.
- Definition E (**color-cyclic**): Vertex permutation, induces cyclic permutation of factors.

Color-cyclic Decompositions

- Given a color-cyclic decomposition, the vertices can be arranged in concentric circles so that one factor can be rotated onto each of the other factors. The following algorithm produces all color-cyclic k-decompositions of K_n.

Algorithm

Let σ be a permutation of n vertices whose odd cycles all have length a multiple of k, and whose even cycles all have length a multiple of $2k$, except perhaps for one fixed point. For each edge orbit, select some edge, and every edge that is a multiple of k steps from it. Let G_0 be the graph induced by the edges, and G_i be the graph produced by applying σ to G_0 i times.

Theorem

A k-decomposition is color-cyclic, with permutation σ, if and only if it can be produced by the algorithm.

- A decomposition of a complete graph into two copies of a self-complementary graph must be color-cyclic, since each factor is mapped to the other.
- The theorem implies a characterization of the structure of self-complementary graphs.

Corollary

(Ringel [1963], Sachs [1962]) *A graph G is self-complementary if and only if there is a permutation σ of $V(G)$ so that each cycle of σ has length a multiple of 4, except for at most one fixed point, and G contains alternate edges along each edge orbit.*

Summary of Definitions

- Definition A: Cyclic vertex permutation, factors map to factors
- Definition B: Cyclic vertex permutation, factors map to factors (distinct unless applied $2n+1$ times)
- Definition C (purely cyclic): Cyclic vertex permutation, induces cyclic permutation of factors.
- Definition D: Vertex permutation, induces cyclic permutation of factors (order n).
- Definition E (color-cyclic): Vertex permutation, induces cyclic permutation of factors.
- To summarize, let A, B, C, D, E be sets of decompositions defined by Definitions A-E.
- Then $B \subseteq A$ and $D \subseteq C \subseteq E$.

Factor-Preserving Decompositions

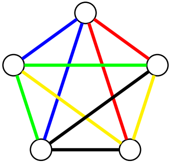
- Perhaps we could define an even more general "cyclic" decomposition.
- Definition F: Vertex permutation, induces permutation of factors.
- (Both permutations can have multiple cycles.)
- Thus $B \subseteq A \subseteq F$ and $D \subseteq C \subseteq E \subseteq F$.
- Let's call this a **factor-preserving decomposition**.

Conjecture

A factor-preserving decomposition of K_{2m+1} into copies of a graph G with size m exists if and only if G has a ρ-labeling.

Non-cyclic Decompositions

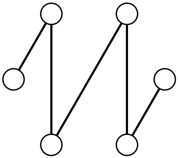
- A decomposition of K_{2m+1} into copies of a graph G with size m does not have to be cyclic.
- Below $K_5 \to \{5\,[P_3]\}$, and the centers of the red and blue factors are the same vertex.

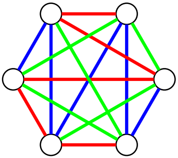


- This generalizes to examples for all larger m.

Non-cyclic Decompositions

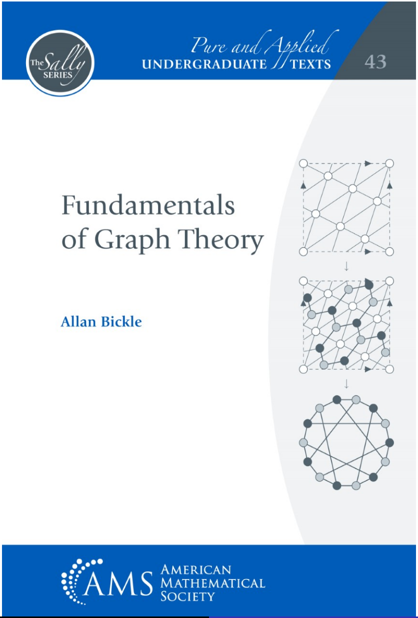
- There are also decompositions that are not factor-preserving.
- (AMSS [1988]) In fact, $K_6 \to \{3[P_6]\}$ in two ways, one cyclic and one not factor-preserving.





- The centers of the paths on the right induce P_4.
- (AMSS [1988]) Also, $K_{10} \to \{9[K_4 - e]\}$, and the decomposition is not factor-preserving.

Thank You!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