k-Degenerate Graphs

Allan Bickle

Western Michigan University

Date

Definition

The k-core of a graph G is the maximal induced subgraph $H \subseteq G$ such that $\delta(H) \geq k$.
The core number of a vertex, $C(v)$, is the largest value for k such that $v \in C_k(G)$.
The maximum core number of a graph, $\widehat{C}(G)$, is the maximum of the core numbers of the vertices of G.
If the maximum core number and minimum degree of G are equal, $\widehat{C}(G) = \delta(G)$, we say G is k-monocore.

Basics

The k-core algorithm (sketch)

Input: graph G with adjacency matrix A, integer k, degree array D.
Recursion: Delete all vertices with degree less than k from G.
(That is, make a list of such vertices, zero out their degrees, and decrement the degrees of their neighbors.)
Result: The vertices that have not been deleted induce the k-core.

Basics

Theorem

Applying the k-core algorithm to graph G yields the k-core of G, provided it exists.

Definition

A vertex deletion sequence of a graph G is a sequence that contains each of its vertices exactly once and is formed by successively deleting a vertex of smallest degree.

Basics

Definition

A graph is k-degenerate if its vertices can be successively deleted so that when deleted, each has degree at most k. The degeneracy of a graph is the smallest k such that it is k-degenerate.

- Thus the k-core algorithm implies a natural min-max relationship.

Corollary

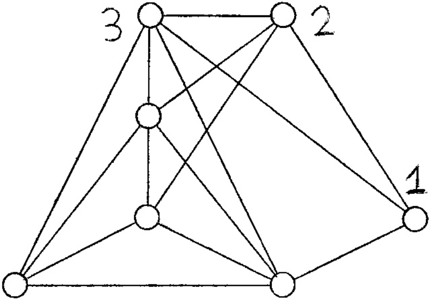
For any graph, its maximum core number is equal to its degeneracy.

Maximal k-degenerate Graphs

Definition

A maximal k-degenerate graph G is a graph that is k-degenerate and is maximal with respect to this property. That is, no more edges can be added to G without creating a $k+1$-core.

- k-degenerate graphs were introduced in 1970 by Lick and White.
- Maximal k-degenerate graphs are the upper extremal k-monocore graphs.



A maximal 3-degenerate graph, which can be seen by successively deleting 1, 2, and 3.

Maximal k-degenerate Graphs

Theorem

The size of a maximal k-degenerate with order n is $k \cdot n - \binom{k+1}{2}$.

Proof.

If G is k-degenerate, then its vertices can be successively deleted so that when deleted they have degree at most k. Since G is maximal, the degrees of the deleted vertices will be exactly k until the number of vertices remaining is at most k. After that, the $n - j^{th}$ vertex deleted will have degree j. Thus the size m of G is

$$m = \sum_{i=0}^{k-1} i + \sum_{i=k}^{n-1} k = \frac{k(k-1)}{2} + k(n-k) = k \cdot n - \binom{k+1}{2}$$

□

Maximal k-degenerate Graphs

Theorem

[Lick/White 1970] Let G be a maximal k-degenerate graph of order n, $1 \leq k \leq n-1$. Then

a. G contains a $k+1$-clique and for $n \geq k+2$, G contains $K_{k+2} - e$ as a subgraph.

b. For $n \geq k+2$, G has $\delta(G) = k$, and no two vertices of degree k are adjacent.

c. G has connectivity $\kappa(G) = k$.

d. Given r, $1 \leq r \leq n$, G contains a maximal k-degenerate graph of order r as an induced subgraph. For $n \geq k+2$, if $d(v) = k$, then G is maximal k-degenerate if and only if $G - v$ is maximal k-degenerate.

e. G is maximal 1-degenerate if and only if G is a tree.

Maximal k-degenerate Graphs

Corollary

Let G be a maximal k-degenerate graph of order n, $1 \le k \le n-1$. Then

a. For $k \ge 2$, the number of nonisomorphic maximal k-degenerate graphs of order $k+3$ is 3.

b. G is k-monocore.

c. G has edge-connectivity $\kappa'(G) = k$, and for $k \ge 2$, an edge set is a minimum edge cut if and only if it is a trivial edge cut.

d. The number of maximal k-degenerate subgraphs of order $n-1$ is equal to the number of vertices of degree k in G that are in distinct automorphism classes.

Proof.

a. $K_{k+2} - e$ is the unique maximal k-degenerate graph of order $k+2$. It has two automorphism classes of vertices, one with two, one with k. Thus there are three possibilities for order $k+3$.

b. G has minimum degree k, and is $k+1$-core-free.

c. First, $k = \kappa(G) \leq \kappa'(G) \leq \delta(G) = k$. Certainly the edges incident with a vertex of minimum degree form a minimum edge cut. The result holds for K_{k+1}. Assume the result holds for all maximal k-core-free graphs of order r, and let G have order $r+1$, $v \in G$, $d(v) = k$, $H = G - v$. Let F be a minimum edge cut of G. If $F \subset E(H)$, the result holds. If F is a trivial edge cut for v, the result holds. If F contained edges both from H and incident with v, it would not disconnect H and would not disconnect v from H.

d. Deleting any minimum degree vertex yields such a subgraph, and deleting any other vertex destroys maximality. The subgraphs will be distinct unless two minimum degree vertices are in the same automorphism class.

Degree Sequences

Lemma

Let G be maximal k-degenerate with order n and nonincreasing degree sequence $d_1, \ldots, d_n$. Then $d_i \leq k + n - i$.

Proof.

Assume to the contrary that $d_i > k + n - i$ for some i. Let H be the graph formed by deleting the $n - i$ vertices of smallest degree. Then $\delta(H) > k$, so G has a $k + 1$-core. □

Lemma

Let G be maximal k-degenerate with degree sequence $d_1 \geq \ldots \geq d_n = k$. Then G has at most $k+1$ vertices whose degrees are equal to the upper bound $\min\{n-1, k+n-i\}$, one of which is v_n, and has exactly $k+1$ such vertices if and only if v_n has the other k as its neighborhood.

Proof.

If G had more than $k+1$ such vertices, then $H = G - v_n$ would have a vertex with degree more than than the maximum possible. If G has exactly $k+1$, then all but v_n must have degree reduced by exactly one in H when v_n is deleted. $\square$

- We can generalize the characterization of degree sequences of trees to maximal k-degenerate graphs.

Theorem

A nonincreasing sequence of integers $d_1, \ldots, d_n$ is the degree sequence of a maximal k-degenerate graph G if and only if $k \leq d_i \leq min\{n-1, k+n-i\}$ and $\sum d_i = 2\left[k \cdot n - \binom{k+1}{2}\right]$ for $0 \leq k \leq n-1$.

Degree Sequences

Proof.

Let $d_1, \ldots, d_n$ be such a sequence.
($\Rightarrow$) Certainly $\triangle(G) \leq n-1$. The other three conditions have already been shown.
($\Leftarrow$) For $n = k+1$, the result holds for $G = K_{k+1}$. Assume the result holds for order r. Let $d_1, \ldots, d_{r+1}$ be a nonincreasing sequence that satisfies the given properties. Let $d_1', \ldots, d_r'$ be the sequence formed by deleting d_{r+1} and decreasing k other numbers greater than k by one, including any that achieve the maximum. (There are at most k by the preceding lemma.) Then the new sequence satisfies all the hypotheses and has length r, so it is the degree sequence for some maximal k-degenerate graph H. Add vertex v_{r+1} to H, making it adjacent to the vertices with degrees that were decreased for the new sequence. Then the resulting graph G has the original degree sequence and is maximal k-degenerate. $\square$

Theorem

Let G be maximal k-degenerate with $\triangle(G) = r$, $n \geq k+1$, and n_i the number of vertices of degree i, $k \leq i \leq r$. Then

$$k \cdot n_k + (k-1)\,n_{k+1} + \ldots + n_{2k-1} = n_{2k+1} + \ldots + (r-2k)\,n_r + k\,(k+1)$$

Proof.

Assume the hypothesis, and let G have order n, size m. Then $\sum_{i=1}^{r} n_i = n$ and

$$\sum_{i=k}^{r} i \cdot n_i = 2m = 2\left[k \cdot n - \binom{k+1}{2}\right] = (2k)\sum_{i=k}^{r} n_i - k\,(k+1).$$

Thus

$$\sum_{i=k}^{r} (i-2k)\,n_i + k\,(k+1) = 0.$$

Corollary

If T is a tree,

$$n_1 = n_3 + 2n_4 + 3n_5 + \ldots + (r-2)\, n_r + 2.$$

For $k = 2$,

$$2n_2 + n_3 = n_5 + 2n_6 + 3n_7 + \ldots + (r-4)\, n_r + 6.$$

- We can bound the maximum degree of a maximal k-degenerate graph. Intuitively, since there are approximately $k \cdot n$ edges in G, its maximum degree should be at least $2k$, provided that G has order large enough to overcome the constant $\binom{k+1}{2}$ subtracted from the size.

Theorem

[Filakova, Mihok, and Semanisin 1997] If G is maximal k-degenerate with $n \geq \binom{k+2}{2}$, then $\triangle(G) \geq 2k$.

Further Structural Results

Theorem

A maximal k-degenerate graph G with $n \geq k+2$ has $2 \leq diam(G) \leq \frac{n-2}{k} + 1$.
If the upper bound is an equality, then G has exactly two vertices of degree k and every diameter path has them as its endpoints.

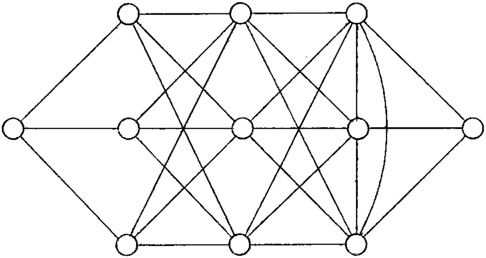


Figure: A maximal 3-degenerate graph with diameter 4 and $n = 11$.

Proof.

Let G be maximal k-degenerate with $r = diam(G)$. For $n \geq k+2$, G is not complete, so $diam(G) \geq 2$. Now G contains u, v with $d(u,v) = r$. Now G is k-connected, so by Menger's Theorem there are at least k independent paths of length at least r between u and v. Thus $n \geq k(r-1)+2$, so $r \leq \frac{n-2}{k}+1$.
Let the upper bound be an equality, and $d(u,v) = r$. Then $n = k(r-1)+2$, and since there are k independent paths between u and v, all the vertices are on these paths. Thus $d(u) = d(v) = k$. If another vertex w had degree k, then $G-w$ would be maximal k-degenerate with $\kappa(G-w) = k-1$, which is impossible. Thus any other pair of vertices has distance less than r. □

Further Structural Results

Theorem

Let $t_1, \ldots, t_r$ be r positive integers which sum to t. Then a maximal t-degenerate graph can be decomposed into r graphs with degeneracies at most $t_1, \ldots, t_r$, respectively.

Proof.

Consider a deletion sequence of a maximal t-degenerate graph G. When a vertex is deleted, the edges incident with it can be allocated to r subgraphs with at most $t_1, \ldots, t_r$ edges going to the respective subgraphs. Thus the subgraphs have at most the stated degeneracies.

□

Corollary

A maximal k-degenerate graph G can be decomposed into K_k and k trees of order $n-k+1$, which span G/K_k.

Proof.

If $n=k$, $G=K_k$, so let the k trees be k distinct isolated vertices. Build G by successively adding vertices of degree k. Allocate one edge to each of the k trees in such a way that each is connected. To do this, assign an edge incident with a vertex of the original clique to the unique tree containing that vertex. Any other edges can be assigned to any remaining tree, since every tree contains every vertex not in the original clique. □

Corollary

If k is odd, a maximal k-degenerate graph decomposes into k trees of order $n - \frac{k-1}{2}$.

Proof.

Let $k = 2r - 1$. Then K_{2r} can be decomposed into k trees of order $r + 1$. $\square$

Corollary

A maximal 2-degenerate graph has two spanning trees that contain all its edges and overlap on exactly one edge. This 'overlap edge' can be any edge that is the last to be deleted by the k-core algorithm.

Enumeration

- We can prove a bijection between labeled maximal k-degenerate graphs and a certain type of sequence.

Algorithm: Prufer k-code

Input: Labeled maximal k-degenerate graph of order n
Iteration: While more than $k+1$ vertices remain, delete the least-labeled vertex v of degree k, and let A_i be the unordered set of k neighbors of v.

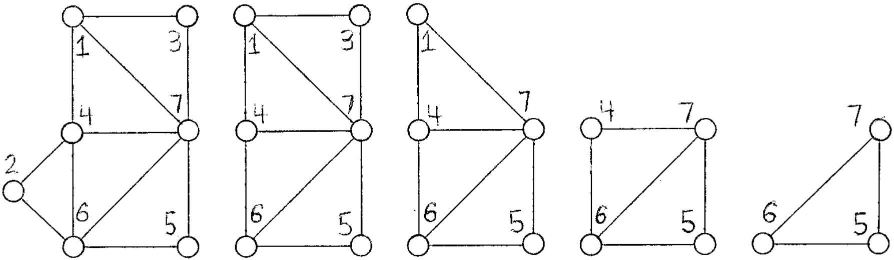


Figure: The Prufer 2-code of this graph is $\{(4,6),(1,7),(4,7),(6,7)\}$.

Enumeration

Theorem

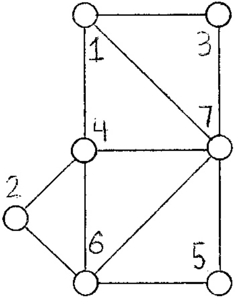
There is a bijection between labeled maximal k-degenerate graphs and Prufer k-codes.

- It is possible to uncode a Prufer k-code.

Enumeration

- Consider uncoding the code $\{(4,6),(1,7),(4,7),(6,7)\}$.
- 2, 3, and 5 do not appear.
- Make 2 adjacent to 4 and 6.
- Make 3 adjacent to 1 and 7.
- Make 1 adjacent to 4 and 7.
- Make 4 adjacent to 6 and 7.
- Add a triangle induced by $\{5,6,7\}$.

Enumeration



Theorem

There are at most $\binom{n}{k}^{n-k-1}$ *labeled maximal k-degenerate graphs of order* n*, with equality exactly when* $k=1$ *or* $n<\frac{k(k+1)}{k-1}$*.*

Proof.

There are $\binom{n}{k}$ different possible sets, and the code contains $n-k-1$ such sets. If there are fewer elements in the sets than vertices, than any code with $n-k-1$ sets of k elements from [n] will yield a graph since there will always be some element that does not appear in any of the sets. This is equivalent to $k(n-k-1)<n$, which gives $k=1$ or $n<\frac{k(k+1)}{k-1}$. However, if $k(n-k-1)\geq n$, any sequence that contains every element of [n] will not yield a graph. □

Corollary

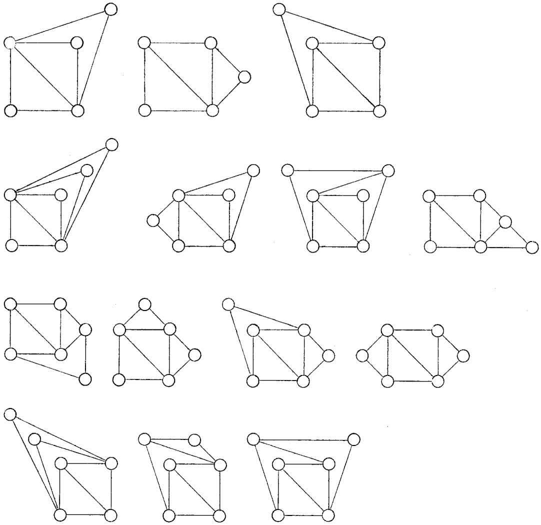
There are exactly $\binom{n}{k}^{n-k-1} - \prod_{t=1}^{n-k-1} \binom{t \cdot k}{k}$ *labeled maximal* k*-degenerate graphs of order* n *when* $n = \frac{k(k+1)}{k-1}$.

Order	3	4	5	6
Number	1	6	100	3285

The number of labeled maximal 2-degenerate graphs.

Order	4	5	6	7
Number	1	3	11	62

The number of unlabeled maximal 2-degenerate graphs.



The maximal 2-degenerate graphs of order 5 and 6.

Definition

A k-tree is a graph that can be formed by starting with K_{k+1} and iterating the operation of making a new vertex adjacent to all the vertices of a k-clique of the existing graph.

Theorem

Every maximal k-degenerate graph G contains a unique k-tree of largest possible order containing a $k+1$-clique that can be used to begin the construction of G.

Proof.

It is obvious that every maximal k-degenerate graph can be constructed beginning with a maximal k-tree. We prove uniqueness. Suppose to the contrary that there is a maximal k-degenerate graph containing two distinct maximal k-trees either of which can be used to begin its construction. Let G be a counterexample of minimum order $n \geq k+3$ containing k-trees T_1 and T_2. Divide the vertices of G into $V(T_1)$, $V(T_2)$, and $S = V(G) - V(T_1) - V(T_2)$. Now G has at least one vertex v of degree k. If $v \in S$, then $G - v$ can be constructed starting with either k-tree, so there is a smaller counterexample. If $v \in V(T_1)$ and $n(T_1) \geq k+2$, then $G - v$ can be still be constructed starting with some other vertex of T_1, so there is a smaller counterexample. If $v \in V(T_i)$, $i \in \{1, 2\}$, and $T_i = K_{k+1}$, then G cannot be constructed starting with T_i since any maximal k-tree that can be used to begin construction of G must contain $K_{k+2} - e$. Thus in any case we have a contradiction. □

k-Trees

- A graph is chordal if it contains no induced cycle of length at least four.
- Each chordal graph has a simplicial elimination ordering. This is a deletion sequence for which the neighbors of each vertex when deleted induce a clique.

Theorem

A graph G is a k-tree $\iff$ G is maximal k-degenerate and G is chordal with $n \geq k+1$.

Proof.

($\Rightarrow$) Let G be a k-tree. G is clearly maximal k-degenerate, since vertices of degree k can be successively deleted until K_{k+1} remains. The construction implies that G has a simplicial elimination ordering, so it is chordal.
($\Leftarrow$) Assume G is maximal k-degenerate and chordal. If $n = k + 1$, it is certainly a k-tree. Assume the result holds for order r, and let G have order $r + 1$. Then G has a vertex v of degree k. The neighbors of v must induce a clique since if v had two nonadjacent neighbors x and y, an $x - y$ path of shortest length in $G - v$ together with yv and vx would produce a cycle with no chord. Thus $G - v$ is a k-tree, hence so is G.

□

Theorem

A maximal k-degenerate graph is a k-tree if and only if it contains no subdivision of K_{k+2}.

Proof.

($\Rightarrow$) Let G be a k-tree. Certainly K_{k+1} contains no subdivision of K_{k+2}. Suppose G is a counterexample of minimum order with a vertex v of degree k. Then $G - v$ is a k-tree with no subdivision of K_{k+2}, so the subdivision in G contains v. But then v is not one of the $k+2$ vertices of degree $k+1$ in the subdivision, so it is on a path P between two such vertices. Let its neighbors on P be u and w. But since the neighbors of v form a clique, $uw \in G - v$, so P can avoid v, implying $G - v$ has a subdivision of K_{k+2}. This is a contradiction. $\square$

Proof.

($\Leftarrow$) Let G be maximal k-degenerate and not a k-tree. Since G is constructed beginning with a k-tree, for a given construction sequence there is a first vertex in the sequence that makes G not a k-tree. Let v be this vertex, and H be the maximal k-degenerate subgraph induced by the vertices of the construction sequence up to v. Then $n(H) \geq k+3$, $d_H(v) = k$, v has nonadjacent neighbors u and w, and $H - v$ is a k-tree. Now there is a sequence of at least two $k+1$-cliques starting with one containing u and ending with one containing w, such that each pair of consecutive $k+1$-cliques in the sequence overlap on a k-clique. Then two of these cliques and a path through v produces a subdivision of K_{k+2}.

$\square$

- Recall that every 3-core contains a subdivision of K_4.

Corollary

[Dirac 1964] If G has $m \geq 2n - 2$, then G contains a subdivision of K_4, and the graphs of size $2n - 3$ that fail to contain a subdivision of K_4 are exactly the 2-trees.

Proof.

Let G have $m \geq 2n - 2 = (3-1)n - \binom{3}{2} + 1$. Hence G contains a 3-core. Then it contains a subdivision of K_4. If a graph of size $2n - 3$ has no 3-core, it is maximal 2-degenerate. By the previous theorem, exactly the 2-trees do not contain a subdivision of K_4. □

- [Mader 1998] If $m \geq 3n - 5$, G contains a subdivision of K_5.

- The chromatic number of a maximal k-degenerate graph is easily determined.

Theorem

If G is maximal k-degenerate with $n \geq k+1$, then $\chi(G) = k+1$.

Proof.

G contains a $k+1$-clique, and and has maximum core number k. □

Coloring k-degenerate Graphs

- Edge coloring is similar to vertex coloring, except that the edges are colored.
- Clearly the edge chromatic number, $\chi_1(G)$, is at least as large as the maximum degree.
- Vizing showed that it is never more than $\triangle(G)+1$.
- A graph is called class one if $\chi(G) = \triangle(G)$, and class two if $\chi(G) = \triangle(G)+1$.

Coloring k-degenerate Graphs

Theorem

[Goufei 2003] Every k-degenerate graph with $\triangle \geq 2k$ is class one.

Corollary

If G is maximal k-degenerate with $n \geq \binom{k+2}{2}$, then G is class one.

- A graph G is overfull if n is odd and $m > \frac{n-1}{2}\triangle(G)$. It is easily seen that an overfull graph is class two.
- This result and the preceding theorem imply that the only maximal 2-degenerate graphs of class two are K_3 and K_4 with a subdivided edge.

Conjecture

A maximal k-degenerate graph is class two $\iff$ it is overfull.

Definition

The edge-arboricity, or simply arboricity $a_1(G)$ is the minimum number of forests into which G can be decomposed.

- If G decomposes into k forests, then $m(G) \leq k \cdot (n(G) - 1)$. Hence this condition applies to any subgraph of G.

Theorem

[Nash-Williams 1964] For every nonempty graph G, $a_1(G) = \max_{H \subseteq G} \left\lceil \frac{m(H)}{n(H)-1} \right\rceil$, where the maximum is taken over all induced subgraphs of G.

- This is a difficult theorem to prove, though a relatively short proof appears in [Chen, Matsumoto, Wang, Zhang, and Zhang 1994].
- This theorem requires determining a maximum over all induced subgraphs of a graph, which is impractical for all but the smallest graphs.

Definition

The a-density of a nontrivial graph G is $\frac{m}{n-1}$.

- The arboricity of maximal k-degenerate graphs was determined in [Patil 1984]. We provide a much shorter proof.
- Note that it follows immediately from an earlier theorem that if G is maximal k-degenerate, then $a_1(G) \leq k$. The arboricity may be smaller if n is small relative to k.

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Theorem

Let G be maximal k-degenerate. Then $a_1(G) = \left\lceil k - \binom{k}{2}\frac{1}{n-1} \right\rceil$.

Proof.

A maximal k-degenerate graph of order n has size $m = k \cdot n - \binom{k+1}{2}$. Then its a-density is $\frac{m}{n-1} = \left[k \cdot n - \binom{k+1}{2}\right]\frac{1}{n-1} = k + \left[k - \binom{k+1}{2}\right]\frac{1}{n-1} = k - \binom{k}{2}\frac{1}{n-1}$. Note that this function is monotone with respect to n. Now any subgraph of a k-degenerate graph is also k-degenerate, so this implies that any proper subgraph of G has smaller a-density. Then by Nash-Williams' theorem, $a_1(G) = \left\lceil k - \binom{k}{2}\frac{1}{n-1} \right\rceil$. □

Definition

Given positive integers $t_1, t_2, \ldots, t_k$, the classical Ramsey number $r(t_1, \ldots, t_k)$ is the smallest integer n such that for any decomposition of K_n into k factors, for some i, the i^{th} factor has a t_i-clique.

- This problem can be modified to require the existence of other classes of graphs.
- Since classical Ramsey numbers are defined, such modifications are also defined, since every finite graph is a subgraph of some clique.

Definition

Given nonnegative integers $t_1, t_2, \ldots, t_k$, the Ramsey core number $rc(t_1, t_2, \ldots, t_k)$ is the smallest n such that for all edge colorings of K_n with k colors, there exists an index i such that the subgraph induced by the i^{th} color, H_i, has a t_i-core.

Theorem

1. $rc(t_1, t_2, \ldots, t_k) \leq r(t_1 + 1, \ldots, t_k + 1)$, the classical multidimensional Ramsey number.

2. For any permutation σ of [k], $rc(t_1, t_2, \ldots, t_k) = rc\left(t_{\sigma(1)}, t_{\sigma(2)}, \ldots, t_{\sigma(k)}\right)$. Thus we need only consider nondecreasing orderings of the numbers.

3. $rc(0, t_2, \ldots, t_k) = 1$

4. $rc(1, t_2, \ldots, t_k) = rc(t_2, \ldots, t_k)$.

Theorem

For k dimensions, $rc(2,2,\ldots,2) = 2k+1$.

Proof.

It is well known that the complete graph K_{2k} can be decomposed into k spanning paths, each of which has no 2-core. Thus $rc(2,2,\ldots,2) \geq 2k+1$. K_{2k+1} has size $\binom{2k+1}{2} = k(2k+1)$, so if it decomposes into k graphs, one of them has at least $2k+1$ edges, and hence contains a cycle. Thus $rc(2,2,\ldots,2) = 2k+1$. □

Definition

The multidimensional upper bound for the Ramsey core number $rc(t_1, t_2, \ldots, t_k)$ is the function $B(t_1, t_2, \ldots, t_k)$, where $T = \sum t_i$ and

$$B(t_1, .., t_k) = \left\lceil \frac{1}{2} - k + T + \sqrt{T^2 - \sum t_i^2 + (2 - 2k)T + k^2 - k + \frac{9}{4}} \right\rceil$$

Theorem

[The Upper Bound] $rc(t_1, t_2, \ldots, t_k) \leq B(t_1, \ldots, t_k)$.

Proof.

The size of a maximal k-core-free graph of order n is $(k-1)n - \binom{k}{2}$. Now by the Pigeonhole Principle, some H_i has a t_i-core when

$$\binom{n}{2} \geq \sum_{i=1}^{k}\left((t_i-1)n - \binom{t_i}{2}\right) + 1.$$

This leads to

$$n^2 - n \geq 2n\sum_{i=1}^{k}(t_i-1) - \sum_{i=1}^{k}\left(t_i^2 - t_i\right) + 2.$$

Thus we obtain a quadratic inequality $n^2 - bn + c \geq 0$ with $b = 1 + 2\sum t_i - 2k$ and $c = \sum\left(t_i^2 - t_i\right) - 2$.

$\square$

Proof.

By the quadratic formula, $n \geq \frac{1}{2}\left(b + \sqrt{b^2 - 4c}\right)$ and

$$b^2 - 4c = \left(1 + 4T - 4k + 4T^2 - 8kT + 4k^2\right) - \left(4\sum t_i^2 - 4T - 8\right)$$

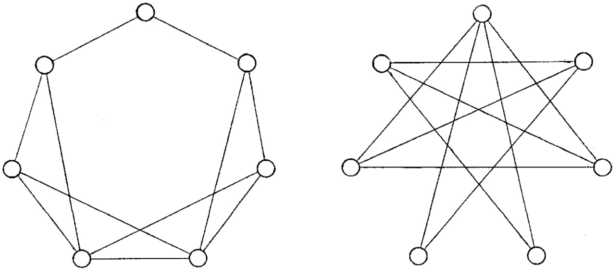
$$= 4\left(T^2 - \sum t_i^2 + (2 - 2k)\,T + k^2 - k + \frac{9}{4}\right)$$

Thus

$$n \geq \left\lceil \frac{1}{2} - k + T + \sqrt{T^2 - \sum t_i^2 + (2 - 2k)\,T + k^2 - k + \frac{9}{4}} \right\rceil.$$

Now $rc\,(t_1, \ldots, t_k) \leq \min\{n \,|\, n \geq B\,(t_1, \ldots, t_k)\} = B\,(t_1, \ldots, t_k)$. □

- Thus to show that a Ramsey core number achieves the upper bound, we must find a decomposition of the complete graph of order $B(t_1, \ldots, t_k) - 1$ for which none of the factors contain the stated cores. For example, this decomposition shows that $rc(3,3) = 8$.



- When first studying this problem in late 2008 I made the following conjecture, initially restricted to two dimensions.

Conjecture

The upper bound is exact. That is, $rc(t_1, t_2, \ldots, t_k) = B(t_1, \ldots, t_k)$.

- To prove this, we state the following theorem due to R. Klien and J. Schonheim from 1992.

Theorem

Any complete graph with order $n < B(t_1, \ldots, t_k)$ has a decomposition into k subgraphs with degeneracies at most $t_1 - 1$, ... , $t_k - 1$.

- The proof of this theorem is long and difficult. It uses a complicated algorithm to construct a decomposition of a complete graph with order satisfying the inequality into k subgraphs given a decomposition of a smaller complete graph into $k-1$ subgraphs without the first $k-1$ cores, a copy of K_{t_k}, and some extra vertices. Thus the proof that the algorithm works uses induction on the number of subgraphs.

Ramsey Core Numbers

Theorem

We have $rc(t_1, t_2, \ldots, t_k) = B(t_1, \ldots, t_k)$.

Proof.

We know that $B(t_1, \ldots, t_k)$ is an upper bound. By the previous theorem, there exists a decomposition of the complete graph of order $B(t_1, \ldots, t_k) - 1$ such that subgraph H_i has degeneracy $t_i - 1$, and hence has no t_i-core. Thus $rc(t_1, t_2, \ldots, t_k) > B(t_1, \ldots, t_k) - 1$, so $rc(t_1, t_2, \ldots, t_k) = B(t_1, \ldots, t_k)$. □

- Since the exact answer depends on a complicated construction, some simpler constructions remain of interest.

Theorem

[The Lower Bound] We have $rc(t_1+1, t_2, \ldots, t_k) \geq rc(t_1, \ldots, t_k) + 1$.

Proof.

Let $n = rc(t_1+1, t_2, \ldots, t_k)$. Then there exists a decomposition of K_{n-1} with each factor having no t_i-core for all i. Let $H = G + v$. Consider the decomposition of K_n formed from the previous decomposition by joining a vertex to the first factor. Then the first factor has no t_1+1-core. Thus $rc(t_1+1, t_2, \ldots, t_k) \geq rc(t_1, \ldots, t_k) + 1$. □

Theorem

Let $t = \binom{r}{2} + q$, $1 \le q \le r$. *Then* $rc(2,t) = \binom{r}{2} + r + q + 1 = t + r + 1 = B(2,t)$.

Proof.

We first show that the Upper Bound for $rc(2,t)$ can be expressed as a piecewise linear function with each piece having slope one and breaks at the triangular numbers. Let $t = \binom{r}{2}$. Let

$$B'(s,t) = s + t - \frac{3}{2} + \sqrt{2(s-1)(t-1) + \frac{9}{4}}.$$

Then $B(s,t) = \lceil B'(s,t) \rceil$. Now $B'(2,t) =$

$$2 + t - \frac{3}{2} + \sqrt{2 \cdot 1(t-1) + \frac{9}{4}} = t + \frac{1}{2} + \sqrt{2\frac{r(r-1)}{2} + \frac{1}{4}} = t + r$$

Proof.

Now $B'(2,t+1) > t+r+1$, so $B(2,t+1) \geq t+r+2$. Then $B(2,t+q) \geq t+r+1+q$ for $q \geq 1$ by the Lower Bound. Now $B'(2,t+r) = B'\left(2,\binom{r+1}{2}\right) = t+r+r+1$, an integer. Thus $B(2,t+r) = t+r+r+1$, so $B(2,t+q) \leq t+r+1+q$ for $1 \leq q \leq r$ by the Lower Bound. Thus $B(2,t+q) = t+r+1+q$, $1 \leq q \leq r$, so $rc(2,t) \leq t+r+1$ for $t = \binom{r}{2}+q$. $\square$

Proof.

We next show that the upper bound is attained with an explicit construction. Let T be a caterpillar whose spine with length r is

$$r - r - (r-1) - (r-2) - \ldots - 4 - 3 - 2,$$

where a number is the degree of a vertex and end-vertices are not shown. Now T has

$$[(r-1) + (r-2) + (r-3) + \ldots + 2 + 1] + 1 = \binom{r}{2} + 1$$

end-vertices, so it has order $n = \binom{r}{2} + r + 1$. The degrees of corresponding vertices in T and $\overline{T}$ must add up to $n - 1 = \binom{r}{2} + r$. □

Proof.

Then the degrees of corresponding vertices in $\overline{T}$ are

$$\binom{r}{2}, \binom{r}{2}, \binom{r}{2}+1, \binom{r}{2}+2, \ldots, \binom{r}{2}+r-3, \binom{r}{2}+r-2.$$

Take the $\left(\binom{r}{2}+1\right)$-core of $\overline{T}$. The first two vertices will be deleted by the k-core algorithm. The p^{th} vertex will be deleted because it has degree $\binom{r}{2}+p-2$ and is adjacent to the first $p-2$ vertices, which were already deleted. Thus all the spine vertices will be deleted, leaving $\binom{r}{2}+1$ vertices, which must also be deleted. Thus $\overline{T}$ has no $\left(\binom{r}{2}+1\right)$-core, and T has no 2-core. Thus $rc\left(2, \binom{r}{2}+1\right) \geq \binom{r}{2}+r+1+1$. Thus $rc\left(2, \binom{r}{2}+q\right) \geq \binom{r}{2}+r+1+q$ by the Lower Bound. Thus $rc\,(2,t) = t+r+1$ for $t = \binom{r}{2}+q$, $1 \leq q \leq r$.

Ramsey Core Numbers

s\t	1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	9	10	11	12
2	3	5	6	8	9	10	12	13	14	15	17
3	4	6	8	10	11	13	14	15	17	18	19
4	5	8	10	11	13	15	16	18	19	20	22
5	6	9	11	13	15	16	18	20	21	23	24
6	7	10	13	15	16	18	20	21	23	25	26
7	8	12	14	16	18	20	22	23	25	26	28

Values of some 2-dimensional Ramsey core numbers.

Thank you!