Nordhaus-Gaddum Theorems for k-Decompositions

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A Motivating Problem

Consider the following problem. An international round-robin sports tournament is held between n teams. The games are split between k locations in different countries, which can host multiple games simultanously. The teams can travel to different locations to play, but it is impractical for the fans to visit more than one location. In this situation, it is reasonable to want teams that play at a given location to play as many games there as possible so that local fans can see them as much as possible. More precisely, we can compute the minimum number of games played by the teams at that location. We then wish to maximize the sum of these minimum numbers over all the locations in the tournament.

Nordhaus-Gaddum Class Theorems

- One common way to study a graph parameter $p(G)$ is to examine the sum $p(G)+p(\overline{G})$ and product $p(G)\cdot p(\overline{G})$.
- Nordhaus and Gaddum proved the following Theorem for chromatic number in 1956.

Theorem

$$2\sqrt{n} \leq \chi(G)+\chi(\overline{G}) \leq n+1$$

$$n \leq \chi(G)\cdot\chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2$$

- Finck [1968] showed that all of these bounds are sharp and determined the extremal graphs.

Nordhaus-Gaddum Class Theorems

- Chartrand and Mitchem [1971] found similar bounds for other graph parameters.
- A theorem providing sharp upper and lower bounds for this sum and product is known as a theorem of the Nordhaus-Gaddum class.
- A graph and its complement decompose a complete graph. Hence a natural generalization of this problem is to consider decompositions into more than two factors.

Definition

A k-decomposition of a graph G is a decomposition of G into k subgraphs.

- The sum upper bound has attracted the most attention.

Definition

For a graph parameter p, let $p(k;G)$ denote the maximum of $\sum_{i=1}^{k} p(G_i)$ over all k-decompositions of G.

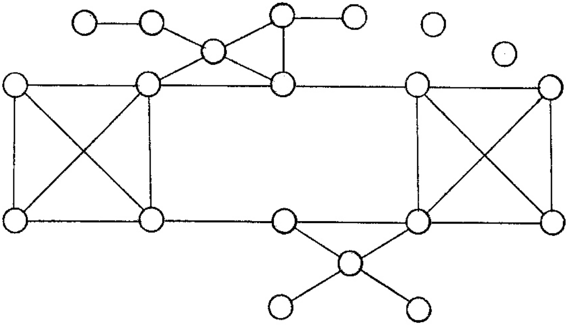
- Furedi, Kostochka, Stiebitz, Skrekovski, and West [2005] explored this upper bound for several different parameters.
- We will tend to describe a k-decomposition as $\{H_1, ..., H_k\}$, where each H_i is a p-critical subgraph.

k-Cores

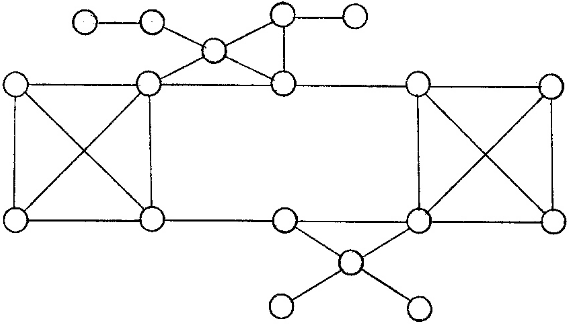
Definition

The k-core of a graph G is the maximal induced subgraph $H \subseteq G$ such that $\delta(H) \geq k$.

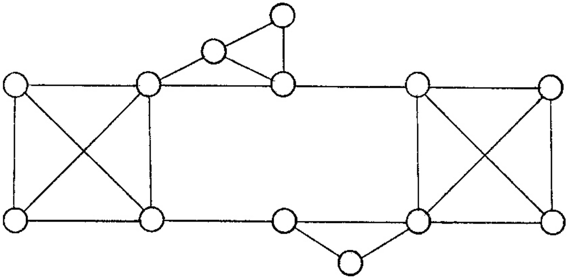
- The k-core was introduced by Steven B. Seidman in a 1983 paper entitled *Network structure and minimum degree*.
- It is immediate that the k-core is well-defined and that the cores are nested.



G is its own 0-core.



The 1-core of G.



The 2-core of G.

The 3-core of G is $2K_4$.

Definition

The core number of a vertex, $C(v)$, is the largest value for k such that $v \in C_k(G)$.
The maximum core number of a graph, $\widehat{C}(G)$, is the maximum of the core numbers of the vertices of G.

- It is immediate that $\delta(G) \leq \widehat{C}(G) \leq \triangle(G)$.

Definition

If the maximum core number and minimum degree of G are equal, $\widehat{C}(G) = \delta(G)$, we say G is k-monocore.

- Translating our motivating problem into graph theory terms, we wish to find $\delta(k; K_n)$ over all values of k and n. We will investigate $\widehat{C}(k; K_n)$, which may be the same thing.

- We need a way to determine the k-core of a graph.

The k-core algorithm (sketch)

Input: graph G with adjacency matrix A, integer k, degree array D.
Recursion: Delete all vertices with degree less than k from G.
(That is, make a list of such vertices, zero out their degrees, and decrement the degrees of their neighbors.)
Result: The vertices that have not been deleted induce the k-core.

Theorem

Applying the k-core algorithm to graph G yields the k-core of G, provided it exists.

Theorem

[Batagelj/Zaversnik 2003] The k-core algorithm has efficiency $O(m)$. (That is, it is linear on the size m.)

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Theorem

a. *We have* $\widehat{C}(G)+\widehat{C}(\overline{G})\leq n-1$.
b. *The graphs for which* $\widehat{C}(G)+\widehat{C}(\overline{G})=n-1$ *are exactly the graphs constructed by starting with a regular graph and iterating the following operation.*
Given $k=\widehat{C}(G)$, H *a* k*-monocore subgraph of* G, *add a vertex adjacent to at least* $k+1$ *vertices of* H, *and all vertices of degree* k *in* H *(or similarly for* $\overline{G}$*).*

Proof.

a. Let $p=\widehat{C}(G)$ and suppose $\overline{G}$ has an $n-p$-core. These cores use at least $(p+1)+(n-p+1)=n+2$ vertices, and hence share a common vertex v. But then $d_G(v)+d_{\overline{G}}(v)\geq p+(n-p)=n$, a contradiction. □

Proof.

b. ($\Leftarrow$) If G is regular with $k = \widehat{C}(G)$, then $\overline{G}$ is $n-k-1$-regular, so $\widehat{C}(G) + \widehat{C}\left(\overline{G}\right) = n-1$. If a vertex v is added as in the operation, producing a graph H, a $k+1$-core is produced, so $\widehat{C}(H) + \widehat{C}\left(\overline{H}\right) = (n+1)-1$.

($\Rightarrow$) Suppose that for a graph G, $\widehat{C}(G) + \widehat{C}\left(\overline{G}\right) = n-1$. If G and $\overline{G}$ are both monocore, then they must be regular. If G has a vertex v that is not contained in the maximum cores of both G and $\overline{G}$, then $\widehat{C}(G-v) + \widehat{C}\left(\overline{G}-v\right) = (n-1)-1$. Then v is contained in the maximum core of one of them, say G. Further, given $k = \widehat{C}(G)$, v is contained in a k-monocore subgraph H of G, and $H-v$ must be $k-1$-monocore. Then v must have been adjacent to all vertices of degree $k-1$ in $H-v$. Thus G can be constructed as described using the operation. □

Maximum Core Number and 2-Decompositions

Corollary

[Nordhaus-Gaddum] We have $\chi(G) + \chi(\overline{G}) \leq n + 1$.

Proof.

We have
$\chi(G) + \chi(\overline{G}) \leq 1 + \widehat{C}(G) + 1 + \widehat{C}(\overline{G}) \leq n - 1 + 2 = n + 1.$ □

- This theorem says that in any extremal 2-decomposition into spanning factors, they must be regular. This generalizes to k-decompositions.

Corollary

Let D be a k-decomposition of K_n into factors that are critical with respect to a maximum core number. Then $\sum_D \left(\widehat{C}\left(G_i\right)\right) \leq n-1$ with equality exactly for decompositions into k spanning regular graphs.

Proof.

Given vertex v, we have $\sum_D \left(\widehat{C}\left(G_i\right)\right) \leq \sum d_{G_i}\left(v\right) \leq n-1$. Equality holds exactly when every factor is regular. □

The Two Factor Theorem

- Next we consider k-decompositions with the restriction that each vertex is contained in exactly two factors. Consider the following construction.

Algorithm

Let r_1, ..., r_k be nonnegative integers at most one of which is odd. Let G_{ij}, $1 \leq i < j \leq k$ be an r_i-regular graph of order $r_i + r_j + 1$, and let $G_{ji} = \overline{G}_{ij}$. Let $G_i = \bigoplus_{j, j \neq i} G_{ji}$. Let S_k be the set of all k-decompositions of the form $\{G_1, ..., G_k\}$ constructed in this fashion.

The Two Factor Theorem

Theorem

[Two Factor Theorem] A k-decomposition with order $n > 1$ and every vertex in exactly two factors has $\sum_D \left(\widehat{C}(G_i)\right) \leq \left(\frac{2k-3}{k-1}\right) n - \frac{k}{2}$, and equality holds exactly for those decompositons in the set S_k.

Proof.

Since each vertex is contained in exactly two of the k factors, so we can partition them into $\binom{k}{2}$ distinct classes. Let $H_{ij} = V(G_i) \cap V(G_j)$ and let $n_{ij} = |H_{ij}|$ for $i \neq j$, $n_{ii} = 0$. Hence $n = \sum_{i,j} n_{ij}$. For $v \in H_{ij}$, we have $\widehat{C}(G_i) \leq d_{G_i}(v) \leq d_{G_i[H_{ij}]}(v) + \sum_{t=1}^{k} n_{it}$. Sum for each of the two factors and each of the $\binom{k}{2}$ classes. Then $(k-1)\sum_{i=1}^{k} \widehat{C}(G_i) \leq 2(k-2)\sum_{i,j} n_{ij} + \sum_{i,j,i\neq j}(n_{ij}-1) = (2k-3)n - \binom{k}{2}$, so $\sum_{i=1}^{k} \widehat{C}(G_i) \leq \left(\frac{2k-3}{k-1}\right) n - \frac{k}{2}$. □

The Two Factor Theorem

Proof.

($\Rightarrow$) If this bound is an equality, then all k of the factors must be regular. Let $r_{ij} = d_{G_i[H_{ij}]}(v)$ for $v \in G_i[H_{ij}]$. Also, all edges between two classes sharing a common factor must be in that factor, so it is a join of $k-1$ graphs. A join of graphs is regular only when they are all regular. Now since G_i is regular, its complement must also be regular. But this implies that all the constants r_{ji}, $j \neq i$ are equal. Let r_i be this common value. Then $n_{ij} = r_i + r_j + 1$, so $n = (k-1)\sum r_i + \binom{k}{2}$. This implies that at most one of r_i and r_j is odd, so at most one of all the r_i's is odd.

($\Leftarrow$) Let G_i be a factor of a decomposition D constructed using the algorithm. It is easily seen that G_i is regular of degree $((k-3)r_i + (k-2) + \sum_j r_j)$. Summing over all the factors, we find that $\sum_D \left(\widehat{C}(G_i)\right) = \left(\frac{2k-3}{k-1}\right) n - \frac{k}{2}$. □

Maximum Core Number and 3-Decompositions

- Now consider 3-decompositions. The formula in the following theorem was proven by Furedi et al.

Theorem

We have $\widehat{C}(3;K_n) = \left\lfloor \frac{3}{2}(n-1) \right\rfloor$, and the extremal decompositions that avchieve $\sum_{i=1}^{3} \widehat{C}(G_i) = \frac{3}{2}(n-1)$ all consist of three $\frac{n-1}{2}$-regular graphs. For $n = 1$, $\{K_1, K_1, K_1\}$ is the only extremal 3-decomposition, and for odd order $n > 1$ they are exactly those in the set S_3.

Proof.

Let G_1, G_2, and G_3 be the three factors of an extremal decomposition for $\widehat{C}(3;K_n)$. It is obvious that $\{K_1, K_1, K_1\}$ is the only possibility for $n = 1$, so let $n > 1$. The previous theorem shows that $\sum_{i=1}^{3} \widehat{C}(G_i) \leq \frac{3}{2}(n-1)$. □

Proof.

Now any vertex can be contained in at most two of the three factors, since its degrees in the three graphs sum to at most $n-1$. Now adding a vertex with adjacencies so that it is contained in exactly one of the three factors increases n by one and $\sum_{i=1}^{3} \widehat{C}(G_i)$ by at most one, so this cannot violate the bound. Thus deleting a vertex of an extremal decomposition contained in only one of the three factors would decrease n by one and $\sum_{i=1}^{3} \widehat{C}(G_i)$ by at most one. For n odd, this is a contradiction and for n even it can occur only when it is the only such vertex.

If there are only two distinct classes, then add a vertex joined to all the vertices of the two disjoint factors. This increases n by one and $\sum_{i=1}^{3} \widehat{C}(G_i)$ by two. Hence if the new decomposition satisfies the bound, so does the original, and if the orginal decomposition attains the bound, then n must be even. □

Proof.

Thus by the Two Factor Theorem, those decompositions with $\sum_{i=1}^{3} \widehat{C}(G_i) = \frac{3}{2}(n-1)$ are exactly those in S_3. Further, by the proof of this theorem the factors of such a decomposition are all $1 + \sum r_j$-regular. Now $2(r_1 + r_2 + r_3) = \sum (n_{ij} - 1) = n - 3$, so $\sum_j r_j = \frac{n-3}{2}$. Thus the factors are all $\frac{n-1}{2}$-regular.
Finally note that joining a vertex to all vertices of one factor of an extremal decomposition of odd order attains the bound for even order, so $\widehat{C}(3; K_n) = \left\lfloor \frac{3}{2}(n-1) \right\rfloor$ for even orders as well. □

- An extremal decomposition of even order can be formed from one of odd order by either joining a vertex to one of the factors or deleting a vertex contained in two factors. However, the decomposition $\{K_4, C_4, C_4\}$ shows that not all extremal decompositions of even order can be formed this way.

- Furedi et al also proved that $\widehat{C}(4;K_n) = \left\lfloor \frac{5}{3}(n-1) \right\rfloor$. We use their proof to show that all extremal decompositions with $n = 3r+1 > 1$ can be constructed by the following algorithm.

Algorithm

Let n, r, a, b, c, and s be nonnegative integers with $n = 3r+1$, $a+b+c = s-1$ and a, b, c, even if s is odd. Let G_1, G_2, G_3 be a, b, c-regular graphs, respectively, of order s. Let G_4, G_5, G_6 be $r-s$-regular graphs of orders $r-a$, $r-b$, $r-c$, respectively. Let S be the set of all decompositions of the form $\left\{G_1 + G_4, G_2 + G_5, G_3 + G_6, \overline{G}_3 + \overline{G}_4 + \overline{G}_6\right\}$.

Theorem

We have $\widehat{C}(4;K_n) = \left\lfloor \frac{5}{3}(n-1) \right\rfloor$. For $n = 1$, $\{K_1, K_1, K_1, K_1\}$ is the only extremal 4-decomposition and the extremal decompositions of order $n = 3r+1 > 1$ that avchieve $\sum_{i=1}^{4} \widehat{C}(G_i) = \frac{5}{3}(n-1)$ are exactly those in S.

Proof.

It is obvious that $\{K_1, K_1, K_1, K_1\}$ is the only possibility for $n = 1$, so let $n > 1$. It is easily checked that the decompositions in S exist and achieve the stated sum. Joining a vertex to one of the factors achieves the stated bound for $n = 3r + 2$, and deleting a vertex contained in two of the factors achieves the bound for $n = 3r$.
As in the previous theorem, it is easily shown that no vertex is contained in a single factor or all four factors. If each vertex is contained in exactly two of the four factors, then the Two Factor Theorem says that $\sum_{i=1}^{4} \widehat{C}(G_i) \leq \frac{5}{3}n - 2$. Hence this decomposition is not extremal for $n = 3r + 1$. □

Proof.

Consider an extremal decomposition with a vertex contained in three of the factors. Call these factors 1, 2, and, 3 so that $\widehat{C}(G_1) \le \widehat{C}(G_2) \le \widehat{C}(G_3)$. Let $H_{123} = V(G_1) \cap V(G_2) \cap V(G_3)$ and $H_{i4} = V(G_i) \cap V(G_4)$. Then $\widehat{C}(G_1) + \widehat{C}(G_2) + \widehat{C}(G_3) \le n-1$, so $\widehat{C}(G_1) + \widehat{C}(G_2) \le \frac{2}{3}(n-1)$. Now $\widehat{C}(G_3) + \widehat{C}(G_4) \le n-1$, so $\sum_{i=1}^{4} \widehat{C}(G_i) \le \frac{5}{3}(n-1)$, and $\widehat{C}(4; K_n) = \left\lfloor \frac{5}{3}(n-1) \right\rfloor$.
If $\sum_{i=1}^{4} \widehat{C}(G_i) = \frac{5}{3}(n-1)$, then $\widehat{C}(G_1) + \widehat{C}(G_2) = \frac{2}{3}(n-1)$ and $\widehat{C}(G_3) + \widehat{C}(G_4) = n-1$. The former implies that $\widehat{C}(G_1) = \widehat{C}(G_2) = \widehat{C}(G_3) = \frac{1}{3}(n-1)$. The latter and this imply that $\widehat{C}(G_4) = \frac{2}{3}(n-1)$ and each vertex in $G_i \cap G_4$, $i \in \{1,2,3\}$, is only adjacent to vertices in these two factors. Hence the vertices partition into H_{123} and $H_{i4} = G_i \cap G_4$, $i \in \{1,2,3\}$ whose orders we call n_{123} and n_{i4}, respectively. Furthermore, each of the factors is regular. □

Proof.

Then H_{123} is decomposed into three regular spanning factors whose degrees are even if n_{123} is odd, and the other sets are decomposed into two regular spanning graphs. Let $r_{i,S} = d_{G_i[H_S]}(v)$ for $v \in G_i[H_S]$. Hence $r_{1,123} + n_{14} = r_{1,14} + n_{123}$, $r_{2,123} + n_{24} = r_{2,24} + n_{123}$, and $r_{3,123} + n_{34} = r_{3,34} + n_{123}$. Now since G_4 is regular, so is $\overline{G}_4$. Thus $r_{1,14} = r_{2,24} = r_{3,34}$, so each of the factors is regular of the same degree. Let $r = \frac{1}{3}(n-1)$ be this common value, $s = n_{123}$, so $r - s = r_{1,14} = r_{2,24} = r_{3,34}$. Let $a = r_{1,123}$, $b = r_{2,123}$, and $c = r_{3,123}$, so $a + b + c = s - 1$, $n_{14} = r - a$, $n_{24} = r - b$, and $n_{34} = r - c$. There are no parity problems, so the extremal decomposition can be constructed by the algorithm. □

Bounds for k=5 and 6

- The values of $\widehat{C}(k;K_n)$ for $k \in \{2,3,4\}$ satisfy $\widehat{C}(k;K_n) = \left\lfloor \frac{2k-3}{k-1}(n-1) \right\rfloor$. In fact, Furedi et al produced a simple construction to prove that $\widehat{C}(k;K_n) \geq \left\lfloor \frac{2k-3}{k-1}(n-1) \right\rfloor$, but this is not an equality for $k \geq 5$.

Algorithm

Let S_5^* be the set of all decompositions that can be constructed as follows. Take a decomposition D in S_4 with the additional property that the sum of some two of the four r_i's equals the sum of the other two r_i's (e.g. $r_1 + r_2 = r_3 + r_4$). Let r be this common value. Add the factor $K_{r+1,r+1}$ to the decomposition.

Theorem

We have $\widehat{C}(5; K_n) \geq \left\lfloor \frac{11}{6}n - 2 \right\rfloor$.

Proof.

This construction has $\sum_{i=1}^{5} \widehat{C}(G_i) = \frac{5}{3}n - 2 + \frac{n}{6} = \frac{11}{6}n - 2$ for any order that it can attain. The proof of the Two Factor Theorem shows that a decomposition in S_k has order $n = (k-1)\sum r_i + \binom{k}{2}$. For $k = 4$, this gives $n = 3\sum r_i + 6$. To satisfy the property in the construction, all the r_i's must be even, and it is obvious that any nonnegative even r can be attained. Hence for each positive order $n = 6r$ there is a decomposition in S_5^* with this order. Successively deleting five vertices contained in exactly two factors from such a decomposition provides decompositions attaining the bound for the other five classes of orders mod 6. □

Bounds for k=5 and 6

Conjecture

For $n \geq 2$, $\widehat{C}(5; K_n) = \left\lfloor \frac{11}{6} n - 2 \right\rfloor$.

- The best known upper bound, due to Furedi et al says that $\widehat{C}(5; K_n) \leq 2n - 3$.

Algorithm

Let S_6^* be the set of all decompositions that can be constructed as follows. Take a decomposition D in S_4 with the additional property that two pairs of two of the four r_i's are equal. (e.g. $r_1 = r_2$ and $r_3 = r_4$). Let r be the sum of these two values. Add two copies of the factor $K_{r+1,r+1}$ to the decomposition.

Theorem

For $n \geq 4$, $\widehat{C}(6; K_n) \geq 2n - 2$.

Bounds for k=5 and 6

Proof.

This construction has $\sum_{i=1}^{6} \widehat{C}(G_i) = \frac{5}{3}n - 2 + 2\binom{n}{6} = 2n - 2$ for any order that it can attain. The proof of the Two Factor Theorem shows that a decomposition in S_k has order $n = (k-1)\sum r_i + \binom{k}{2}$. For $k = 4$, this gives $n = 3\sum r_i + 6$. To satisfy the property in the construction, all the r_i's must be even, and any nonnegative even $r = 4s$ can be attained. Hence for each positive order $n = 12s + 6$ there is a decomposition in S_6^* with this order. Successively deleting vertices contained in exactly two factors from such a decomposition provides decompositions attaining the bound when $4 \leq n \leq 6$, $12 \leq n \leq 18$, and $n \geq 20$. Joining a vertex to each of the two disjoint factors when $n = 12s + 6$ works for $n \in \{7, 19\}$. Now $\{2[K_4], 4[C_4]\}$ works for $n = 8$ and $\{3[K_4], 3[K_{3,3}]\}$ works for $n = 10$. Joining a vertex to disjoint factors in these last two works for $n \in \{9, 11\}$. □

Bounds for k=5 and 6

Conjecture

For $n \geq 4$, $\widehat{C}(6; K_n) = 2n - 2$.

- The best known upper bound, due to Furedi et al says that $\widehat{C}(6; K_n) \leq \frac{5}{2}n - \frac{7}{2}$.
- The constructions that we have seen so far start with a small decomposition and 'expand' it to a bigger one. In some cases, this process can be generalized.

Expanded Constructions

Theorem

Suppose there is a k-decomposition of K_n into regular subgraphs and $\sum_{i=1}^{k} \widehat{C}(G_i) = c(n-1)$. Then there are infinitely many other k-decompositions with order n' and $\sum_{i=1}^{k} \widehat{C}(G_i) = c(n'-1)$.

Proof.

Let $r = n-1$. Let D be a decomposition of K_{rt+1} into r t-regular spanning factors, where t is even if r is even. Form a k-decomposition D' with order n' by replacing each vertex of K_n with a copy of D so that if vertex v has degree d_i in G_i, then d_i of the r factors are merged together. Finally, join the corresponding factors in different copies of D.

If the factor G_i has degree d_i in K_n, then the factor G_i' has degree $d_i(rt+1)+d_it$. Now since $\sum_{i=1}^{k} d_i = c(n-1)$ and $n' = n(rt+1)$, $\sum_{i=1}^{k}(d_i(rt+1)+d_it) = (rt+1+t)\sum d_i = (rt+1+t)c(n-1)$ $= c[n(rt+1)-1+t(n-1-r)] = c(n'-1)$. □

- We now consider a number of decompositions that can be expanded to infinite families via the previous theorem.
- Decompose K_n into $k = \binom{n}{2}$ K_2's. Then $\sum_{i=1}^{k} \widehat{C}(G_i) = \binom{n}{2} = \frac{n}{2}(n-1) = \frac{1+\sqrt{1+8k}}{4}(n-1)$. Thus this sum can be achieved for infinitely many orders whenever k is a triangular number.
- Decompose K_n into K_3's, which can occur whenever $n \equiv 1$ or 3 mod 6. Such a decomposition has $k = \frac{1}{3}\binom{n}{2} = \frac{n(n-1)}{6}$ triangles, so $\sum_{i=1}^{k} \widehat{C}(G_i) = 2\frac{n(n-1)}{6} = \frac{n}{3}(n-1) = \frac{1+\sqrt{1+24k}}{6}(n-1)$.
- In particular, consider $k = 7$. Let H be an r-regular graph of order $3r+1$. Let $G = H + H + H$. Then G is $7r+2$-regular, and 7 copies of G form a decomposition of order $n = 7(3r+1) = 21r+7$, so $\frac{n-1}{3} = 7r+2$. Then $\sum_{i=1}^{7} \widehat{C}(G_i) = 7(7r+2) = \frac{7}{3}(n-1)$. This construction shows that $\widehat{C}(7; K_n) \geq \left\lfloor \frac{7}{3}(n-1) \right\rfloor$ for $n = 7(3r+1)$.

Expanded Constructions

- Decompose K_n into K_4's, which can occur whenever $n \equiv 1$ or 4 mod 12 [Hanani 1961]. Such a decomposition has $k = \frac{1}{6}\binom{n}{2} = \frac{n(n-1)}{12}$ K_4's, so $\sum_{i=1}^{k} \widehat{C}(G_i) = 3\frac{n(n-1)}{12} = \frac{n}{4}(n-1) = \frac{1+\sqrt{1+48k}}{8}(n-1)$.
- Decompose K_n into K_5's, which can occur whenever $n \equiv 1$ or 5 mod 20 [Hanani 1975]. Such a decomposition has $k = \frac{1}{10}\binom{n}{2} = \frac{n(n-1)}{20}$ K_5's, so $\sum_{i=1}^{k} \widehat{C}(G_i) = 4\frac{n(n-1)}{20} = \frac{n}{5}(n-1) = \frac{1+\sqrt{1+80k}}{10}(n-1)$.
- Let $n = p^2 + p + 1$, where p is a prime power. Then there is a projective plane with n points and n lines, which correspond to vertices and factors of a decomposition. Then $\sum_{i=1}^{k} \widehat{C}(G_i) = kp = \frac{kp}{k-1}(n-1) = \frac{p^2+p+1}{p+1}(n-1) = \frac{\left(-1+\sqrt{4k-3}\right)k}{2(k-1)}(n-1)$.
- Let $k[G]$ mean that factor G occurs k times in a decomposition.

Expanded Constructions

k	$\sum \widehat{C}(G_i)$	decomposition
2	$n-1$	$\{2[K_1]\}$
3	$\frac{3}{2}(n-1)$	$\{3[K_2]\}$
4	$\frac{5}{3}(n-1)$	$\{K_3, 3[K_2]\}$
5	$\frac{9}{5}(n-1)$	$\{4[K_3], 3K_2\}$
6	$2(n-1)$	$\{6[K_2]\}$
7	$\frac{7}{3}(n-1)$	$\{7[K_3]\}$
8	$\frac{9}{4}(n-1)$	$\{K_3, 7[K_2]\}$
9	$\frac{12}{5}(n-1)$	$\{3[K_3], 6[K_2]\}$
10	$\frac{5}{2}(n-1)$	$\{10[K_2]\}$
11	$\frac{19}{7}(n-1)$	$\{8[K_3], 2[K_2], 2K_2\}$
12	$3(n-1)$	$\{12[K_3]\}$
13	$\frac{13}{4}(n-1)$	$\{13[K_4]\}$
14	$\frac{25}{8}(n-1)$	$\{11[K_3], 3[K_2]\}$
15	$3(n-1)$	$\{15[K_2]\}$
16	$\frac{17}{5}(n-1)$	$\{K_5, 15[K_3]\}$
[illegible]	[illegible]	[illegible]

Expanded Constructions

- There is another way to generate decompositions that are better for some orders. If a decomposition has $\sum_{i=1}^{k} \widehat{C}(G_i) = c(n-1)$, then some factor G_i has $\widehat{C}(G_i) \leq \frac{c}{k}(n-1)$. Generalizing this, we have the following.

Theorem

If there is a decomposition of K_n with $\sum_{i=1}^{k} \widehat{C}(G_i) = c(n-1)$, then given $0 \leq p \leq k-1$, there is a decomposition of K_n with $\sum_{i=1}^{k-p} \widehat{C}(G_i) \geq c\frac{k-p}{k}(n-1)$.

- Furedi et al also proved the general upper bound that for all positive integers n and k, $\widehat{C}(k; K_n) \leq \sqrt{k} \cdot n$. This is not attained for any values of n and k. Using essentially the same approach, this can be strengthened to a sharp bound.

A General Upper Bound

Theorem

For all positive integers n and k, we have $\widehat{C}(k; K_n) \leq -\frac{k}{2} + \sqrt{\frac{k^2}{4} + kn(n-1)}$. *This is an equality exactly when there is a decomposition of* K_n *into k cliques of equal size.*

Proof.

For a k-decomposition, let $d_i = \widehat{C}(G_i)$ and $D = \sum \widehat{C}(G_i)$. Then $m(G_i) \geq \binom{d_i+1}{2}$. Now

$$\frac{n(n-1)}{2} = \binom{n}{2} \geq \sum_{i=1}^{k} \binom{d_i+1}{2} = \frac{1}{2}\sum_{i=1}^{k}(d_i^2 + d_i) \geq \frac{1}{2}\left(\frac{D^2}{k} + D\right).$$

The first inequality is attained exactly when all the factors are cliques, and the second is attained exactly when all the cliques have the same size. Hence $kn(n-1) \geq D^2 + kD$, so $D^2 + kD - kn(n-1) \leq 0$, and $D \leq -\frac{k}{2} + \sqrt{\frac{k^2}{4} + kn(n-1)}$. □

- We can obtain the successively simpler but weaker formulas $\widehat{C}\left(k;K_n\right) \leq -\frac{k}{2} + \sqrt{\frac{k^2}{4} + kn\left(n-1\right)} < \sqrt{kn\left(n-1\right)} < \sqrt{k}\left(n-\frac{1}{2}\right) < \sqrt{k} \cdot n$ as corollaries. The last is the bound reported by Furedi et al.
- A decomposition of K_n into k cliques of equal size is a block design. In particular, it is a $\left(n, k, \frac{k+\sqrt{k^2+4kn(n-1)}}{2n}, \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{n(n-1)}{k}}, 1\right)$-design. Hence the previous result will attain equality whenever such a design exists.

A General Upper Bound

Corollary

We have

1. $\widehat{C}\left(\binom{n}{2}; K_n\right) = \binom{n}{2}$ *for* $n \geq 2$
2. $\widehat{C}\left(\frac{n(n-1)}{6}; K_n\right) = \frac{n(n-1)}{3}$ *for* $n \equiv 1$ *or 3 mod 6*
3. $\widehat{C}\left(\frac{n(n-1)}{12}; K_n\right) = \frac{n(n-1)}{4}$ *for* $n \equiv 1$ *or 4 mod 12*
4. $\widehat{C}\left(\frac{n(n-1)}{20}; K_n\right) = \frac{n(n-1)}{5}$ *for* $n \equiv 1$ *or 5 mod 20*
5. $\widehat{C}(n; K_n) = \frac{\left(-1+\sqrt{4n-3}\right)n}{2}$ *for* $n = p^2 + p + 1$, *where* p *is a prime power*

Other Parameters

- It is immediate that $\kappa(k, K_n) \leq \lambda(k, K_n) \leq \delta(k, K_n) \leq \widehat{C}(k; K_n)$. Furthermore, the decompositions above show that these are all equalities for $1 \leq k \leq 4$.

Conjecture

For all positive integers n and k, we have $\kappa(k, K_n) = \lambda(k, K_n) = \delta(k, K_n) = \widehat{C}(k; K_n)$.

Other Parameters

But what about chromatic number?

Stay tuned for part two next week!

Thank you!

Chromatic Number

Definition

The chromatic number of a graph, $\chi(G)$, is the smallest number of subsets into which the vertices of a graph can be partitioned so that no two vertices in the same subset are adjacent.
A list coloring of a graph begins with lists of length k assigned to each vertex and chooses a color from each list to obtain a proper vertex coloring. A graph G is k-choosable if any assignment of lists to the vertices permits a proper coloring. The list chromatic number $\chi_l(G)$, is the smallest k such that G is k-choosable.

Chromatic Number

Theorem

[The core number bound] For any graph G,
$\chi(G) \leq \chi_l(G) \leq 1 + \widehat{C}(G)$.

Proof.

Establish a construction sequence for G. Each vertex has degree at most equal to its core number when colored. Coloring it uses at most one more color. Thus $\chi(G) \leq 1 + \widehat{C}(G)$. □

- The bound $\chi(G) \leq 1 + \widehat{C}(G)$. was first proved by Szekeres and Wilf in 1968, stated in different terms. The bounds for list coloring were proved by Erdos, Rubin, and Taylor [1979], who introduced list coloring.
- Certainly we have $\widehat{C}(G) \leq \triangle(G)$ for all graphs. Thus the core number bound is always at least as good as the maximum degree bound.

Proposition

Let G be a connected graph. Then $\widehat{C}(G) = \triangle(G) \iff G$ is regular.

Proof.

If G is regular, then $\delta(G) = \triangle(G)$, so the result is obvious.
For the converse, let $\widehat{C}(G) = \triangle(G) = k$. Then G has a subgraph H with $\delta(H) = \triangle(G) \geq \triangle(H)$, so H is k-regular. If H were not all of G, then since G is connected, some vertex of H would have a neighbor not in H, implying that $\triangle(G) > \triangle(H) = \delta(H) = \triangle(G)$. But this is not the case, so $G = H$, and G is regular. □

Brooks' Theorem

Lemma

Given $r \geq 3$, if G is an r-regular 2-connected noncomplete graph, then G has a vertex v with two nonadjacent neighbors x and y such that $G - x - y$ is connected.

Proof.

Let G satisfy the hypothesis. Let u be a vertex of G. If $G - u$ is 2-connected, let u be x and y be a vertex at distance two from u, which exists because G is regular and not complete. Let v be their common neighbor.

If $G - u$ has connectivity one, then let v be u. Then G has at least two end-blocks, and u has neighbors in all of them. Let x, y be two such neighbors. They must be nonadjacent, and $G - x - y$ is connected since blocks have no cut-vertices and $r \geq 3$. □

Brooks' Theorem

Theorem

[Brooks' Theorem] If G is connected, then $\chi(G) = 1 + \triangle(G) \iff G$ is complete or an odd cycle.

Proof.

Equality certainly holds for cliques and odd cycles. Let G satisfy the hypotheses. Then by the previous result, G is r-regular. The result certainly holds for $r \leq 2$, so we may assume $r \geq 3$. If G had a cut-vertex, each block could be colored with fewer than $r+1$ colors to agree on that vertex, so we may assume G is 2-connected and to the contrary not a clique.

By the lemma, we can establish a deletion sequence for G starting with some vertex v and ending with its nonadjacent neighbors x and y so that all vertices but v have at most $r-1$ neighbors when deleted. Reversing this yields a construction sequence and coloring greedily gives x and y the same color, so G needs at most r colors. □

- Brooks' Theorem extends to list coloring. The proof is slightly more complicated. It is trivial that a graph can be colored to agree on its blocks, but this is more complicated for list coloring.

Lemma

Let G be a connected $r-1$-degenerate graph with a construction sequence ending with a vertex v of order $1 \leq d \leq r-1$. Then for any assignment of lists of length r to the vertices, there are list colorings using at least $d+1$ different colors from the list for v.

- Erdos, Rubin, and Taylor [1979] originally proved Brooks' Theorem for list coloring as part of a more general result with a longer proof.

Theorem

We have $\widehat{C}(G) + \widehat{C}(\overline{G}) \leq n - 1$.

Corollary

[Nordhaus-Gaddum] We have $\chi(G) + \chi(\overline{G}) \leq n + 1$.

Proof.

We have

$$\chi(G) + \chi(\overline{G}) \leq 1 + \widehat{C}(G) + 1 + \widehat{C}(\overline{G}) \leq n - 1 + 2 = n + 1. \qquad \square$$

- We would like to characterize the extremal decompositions for the Nordhaus-Gaddum Theorem. Note that if a 2-decomposition of K_n achieves $\chi(G)+\chi(\overline{G})=n+1$, then we can easily construct a 2-decomposition of K_{n+1} with $\chi(G)+\chi(\overline{G})=n+2$, by joining a vertex to all the vertices of a color-critical subgraph of G or $\overline{G}$, and allocating the extra edges arbitrarily. Conversely, we may be able to delete some vertex v of K_n so that $\chi(G)+\chi(\overline{G})=n$. If this is impossible, we say that an extremal decomposition is fundamental.

Definition

A k-decomposition of K_n with $K=\sum_{i=1}^{k}\chi(G_i)$ achieving the maximum possible such that no vertex v of K_n can be deleted so that $\sum_{i=1}^{k}\chi(G_i-v)=K-1$ is called a fundamental decomposition.

Chromatic Number and 2-Decompositions

Theorem

For $k = 2$, the fundamental decompositions that attain $\chi(2; K_n) = n+1$ are $\{K_1, K_1\}$ and $\{C_5, C_5\}$.

Proof.

It is easily seen that these decompositions satisfy the equation and are fundamental. Consider a fundamental 2-decomposition $\{G, \overline{G}\}$. Then both graphs are connected. Let $\chi(G) = r$, so that $\chi(\overline{G}) = n+1-r$. Then G is an $r-1$-core and $\overline{G}$ is an $n-r$-core. But then G and $\overline{G}$ must be regular, since $n-1 \leq \widehat{C}(G) + \widehat{C}(\overline{G}) \leq d_G(v) + d_{\overline{G}}(v) \leq n-1$. Now by Brooks' Theorem, the only connected regular graphs achieving $\chi(G) = 1 + \widehat{C}(G)$ are cliques and odd cycles. The only such graphs whose complements are connected and also achieve the upper bound are are K_1 and C_5. Thus the fundamental 2-decompositions are as stated. □

Chromatic Number and 2-Decompositions

Corollary

The extremal 2-decompositions for the upper bound of the Nordhaus-Gaddum theorem are exactly $\{K_p, K_{n-p+1}\}$ *and* $\{C_5 + K_p, C_5 + K_{n-p-5}\}$.

Proof.

It is immediate that these are extremal 2-decompositions. Assume that we have an extremal 2-decomposition $\{G, \overline{G}\}$ with order n and let G be r-critical. If the critical subgraphs overlap on a single vertex and $G = K_r$, then $\overline{G} = K_{n-r} + rK_1$, which is uniquely $n-r+1$-colorable. Deleting any edge of the copy of K_{n-r} would reduce the chromatic number, so K_{n-r+1} is the only possible $n-r+1$-critical subgraph. If $G \neq K_r$ has order $p \geq r+2$, then the critical subgraph of $\overline{G}$ is contained in $K_{n-p} + pK_1$, which is impossible. If the critical subgraphs overlap on C_5, the argument is similar. □

Chromatic Number and 2-Decompositions

- In 1968, H. J. Finck determined a similar but inelegant characterization whose proof is more than three pages long. In 2008, Starr and Turner determined the following alternative characterization.

Theorem

Let G and $\overline{G}$ be complementary graphs on n vertices. Then $\chi(G) + \chi(\overline{G}) = n + 1$ if and only if $V(G)$ can be partitioned into three sets S, T, and $\{x\}$ such that $G[S] = K_{\chi(G)-1}$ and $G[T] = K_{\chi(\overline{G})-1}$.

- This characterization leaves something to be desired since it is not obvious which graphs satisfy the condition given in the theorem. However, this result follows immediately as a corollary to the previous theorem.

- We can generalize to k-decompositions into spanning factors.

Corollary

Let D be a k-decomposition of K_n into factors that are critical with respect to chromatic number. Then $\sum_D (\chi(G_i)) \leq n - 1 + k$ with equality exactly for $\{K_n, (k-1)[\overline{K}_n]\}$ and $\{k[C_n]\}$, $n = 2k+1$.

Proof.

We have $\sum_D \chi(G_i) \leq \sum_D \left(1 + \widehat{C}(G_i)\right) \leq n - 1 + k$. By Brooks' Theorem, equality requires that every factor be a clique, empty, or an odd cycle. Thus the extremal decompositions are as stated. □

Chromatic Number and 2-Decompositions

Theorem

If G is regular and $\chi(G) + \chi(\overline{G}) = n$, then the 2-decompositions that satisfy this equation are $\{C_7, \overline{C_7}\}$ and $\{C_4, 2K_2\}$.

Proof.

Assume the hypothesis. Then
$n = \chi(G) + \chi(\overline{G}) \leq 1 + \widehat{C}(G) + 1 + \widehat{C}(\overline{G}) = n + 1$, so exactly one of G or $\overline{G}$ achieves the core number bound, say G. If G is connected, then by Brooks' Theorem, G is a complete graph or odd cycle. But the complement of a complete graph also achieves the upper bound. If G is an odd cycle of length at least 5, then $\chi(\overline{C}_n) = \frac{n+1}{2}$. But $\overline{C}_n$ is $n-3$-regular, so $\frac{n+1}{2} = n - 3$ implies $n = 7$. □

Proof.

If G is disconnected, then it is a union of r-regular components, at least one of which is a clique or an odd cycle. Consider starting with only this component and adding another component with order k. This increases $\chi(G) + \chi(\overline{G})$ by at most $k - r$. Thus to satisfy the equation we want $r = 1$, so the new component is K_2, and no other component can be added. Thus only the 2-decomposition $\{C_4, 2K_2\}$ works. □

Corollary

[Nordhaus-Gaddum] We have $\chi_l(G) + \chi_l(\overline{G}) \leq n + 1$.

Proof.

We have

$\chi_l(G) + \chi_l(\overline{G}) \leq 1 + \widehat{C}(G) + 1 + \widehat{C}(\overline{G}) \leq n - 1 + 2 = n + 1.$ □

List Coloring and 2-Decompositions

Theorem

For $k = 2$, the fundamental decompositions that attain $\chi_l(2; K_n) = n + 1$ are $\{K_1, K_1\}$ and $\{C_5, C_5\}$.

Corollary

The extremal 2-decompositions for the upper bound of the Nordhaus-Gaddum theorem are exactly $\{K_p, K_{n-p+1}\}$ and $\{C_5 + K_p, C_5 + K_{n-p-5}\}$.

- The proofs are virtually identical to those for chromatic number.

Chromatic Number and 3-Decompositions

- Jan Plesnik studied $\chi(k; K_n)$ and in 1978 made the following conjecture.

Conjecture [Plesnik's Conjecture]

For $n \geq \binom{k}{2}$, $\chi(k; K_n) = n + \binom{k}{2}$.

- For $k = 2$, this is just the Nordhaus-Gaddum Theorem. Plesnik proved the conjecture for $k = 3$ and determined an upper bound for $\chi(k, K_n)$.
- There is a simple construction that shows $\chi(k; K_n)$ is at least $n + \binom{k}{2}$. Take the line graph $L(K_k)$ with order $\binom{k}{2}$ and decompose it into k copies of K_{k-1}. For any additional vertex, make it adjacent to all the vertices of one of the cliques in the decomposition and allocate any extra edges arbitrarily.

- Plesnik proved a recursive upper bound of $\chi(k;K_n) \leq n + t(k)$, where $t(2) = 1$ and $t(k) = \sum_{i=2}^{k-1} \binom{k}{i} t(i)$.
- Thus $t(3) = 3$ and $t(4) = 18$.
- This implies a worse explicit bound of $\chi(k;K_n) \leq n + 2^{\binom{k+1}{2}}$.
- In 1985, Timothy Watkinson improved this upper bound to $\chi(k;K_n) \leq n + \frac{k!}{2}$.
- In 2005, Furedi, Kostochka, Stiebitz, Skrekovski, and West proved an improved upper bound for large k of $\chi(k;K_n) \leq n + 7^k$.
- All of these bounds remain far from Plesnik's conjecture, however.

- We can describe many fundamental decompositions for $k \geq 3$ using the following construction.

Algorithm [Construction of fundamental k-decompositions]

For $k \geq 3$ and $n \geq \binom{k}{2}$, construct a decomposition of K_n as follows.
1. Start with the line graph $L(K_k)$ decomposed into k copies of K_{k-1}.
2. Replace each vertex by either K_1 decomposed into $\{K_1, K_1\}$ or K_5 decomposed into $\{C_5, C_5\}$.
3. Join each factor to the other factors corresponding to the same copy of K_{k-1} in the decomposition of $L(K_k)$.
4. Allocate any remaining edges arbitrarily.

- We will see below that the graphs produced by this algorithm attain the bound of Plesnik's conjecture. This algorithm produces all such graphs for $k = 2$ but not all for $k = 3$.

Lemma

1. For $k \geq 3$, let D be a k-decomposition with every vertex contained in exactly two color-critical subgraphs of the decomposition that maximizes $\sum_{i=1}^{k} \chi(G_i)$. Then $\sum_{i=1}^{k} \chi(G_i) = n + \binom{k}{2}$.
2. The k-decompositions produced by the preceding algorithm satisfy $\sum_{i=1}^{k} \chi(G_i) = n + \binom{k}{2}$.

Chromatic Number and 3-Decompositions

Proof.

Assume the hypothesis and let H_i be the critical subgraphs of the k graphs. Thus we can partition the n vertices into $\binom{k}{2}$ classes: $V_{ij} = V(H_i) \cap V(H_j)$. Now the edges between V_{ij} and V_{il} may as well be in H_i since this is the only critical subgraph with vertices in both classes. Similarly, if V_{ij} and V_{lm} have no common indices, then no edges between them are contained in a critical subgraph. Then $\chi(H_i) \leq \sum_j \chi(H_i[V_{ij}])$, where $1 \leq j \leq k$, $i \neq j$. Then $n + \binom{k}{2} \leq \sum \chi(H_i) \leq \sum_{i,j} \chi(H_i[V_{ij}]) \leq \sum(n(V_{ij}) + 1) = n + \binom{k}{2}$, with the last inequality following from the Nordhaus-Gaddum theorem. But then we have equalities, which implies that $\sum_{i=1}^{k} \chi(G_i) = n + \binom{k}{2}$, and the two graphs that decompose $K_n[V_{ij}]$ form an extremal 2-decomposition. Since $\{K_1, K_1\}$ and $\{C_5, C_5\}$ are fundamental 2-decompositions, the algorithm produces fundamental k-decompositions. □

Theorem

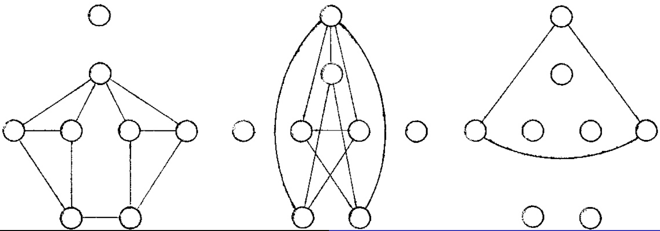
For $k = 3$ and $n \geq 3$, $\chi(3; K_n) = n + 3$.

Proof.

Assume that some fundamental decomposition of K_n into three factors yields $\chi(G_1) = a+1$, $\chi(G_2) = b+1$, and $\chi(G_3) = c+1$, with $a+b+c = n$. We may consider the critical subgraphs H_i of the three graphs, which are a, b, and c-cores, respectively. Now no vertex of K_n can be contained in all three of the H_i's, since this would imply that K_n has at least $a+b+c+1 = n+1$ vertices. Since deleting a vertex from a k-critical graph produces a $k-1$-chromatic graph and the decomposition is fundamental, every vertex is contained in exactly two of the three critical subgraphs. Then by the lemma, $\chi(3; K_n) = n+3$. □

- All the critical subgraphs of the fundamental decompositions must be contained in joins of K_n, $\overline{K}_n$, and C_5.

- The fundamental 3-decompositions produced by the algorithm are $\{K_2, K_2, K_2\}$, $\{W_5, W_5, K_2\}$, $\{W_5, W_5, C_5 + C_5\}$, and $\{C_5 + C_5, C_5 + C_5, C_5 + C_5\}$. However, these are not all the fundamental 3-decompositions. This is because the extremal 2-decompositions produced in the next-to-last sentence of the proof of the lemma need not be fundamental, as can be seen in the figure below.



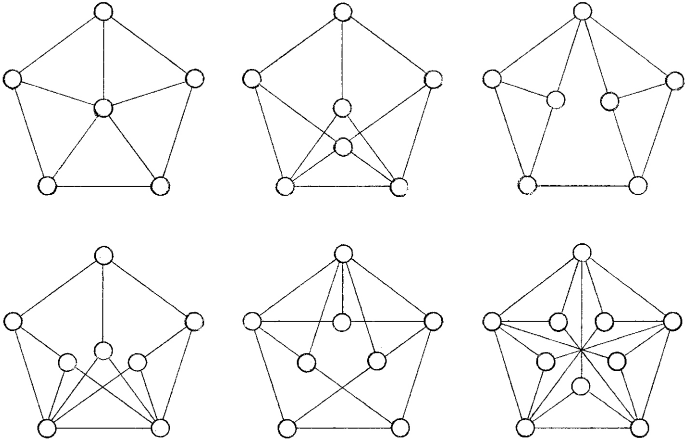
- The joins in question will themselves be color-critical except in the case of $C_5 + \overline{K}_{n-5}$.

Lemma

There are exactly six 4-critical subgraphs of $G = C_5 + \overline{K}_{n-5}$.

- This is proved by a case-checking argument.

Chromatic Number and 3-Decompositions



The 4-critical graphs W_5, $G_1 = M(K_3)$, G_2, G_3, G_4, G_5.

Proof.

Clearly any such subgraph must contain the copy of C_5 and at least one more vertex. If $n = 6$, $W_5 = C_5 + K_1$ is clearly the only possibility. Let S be the set of vertices not on C_5. Since we want a 4-critical subgraph, consider adding each vertex in S one at a time. Then each of them must successively restrict the possible colorings of G, since otherwise it could be deleted and there would be a smaller 4-critical subgraph.

Suppose a vertex in S has degree 4. Note that any vertex of degree 3 either (A) neighbors three consecutive vertices on the cycle or (B) exactly two consecutive vertices on the cycle and a third not adjacent to either of them. Adding a vertex of degree 4 or a degree 3 vertex of type A adjacent to the remaining vertex on the cycle produces a 4-chromatic graph, but at least one edge can be deleted to obtain a 4-critical graph G_1. Checking the possible placements of a degree 3 vertex of type B, only one possibility produces a 4-critical graph. (This is $M(K_3)$, applying Mycielski's construction.) □

Chromatic Number and 3-Decompositions

Proof.

Now suppose that we start with a vertex v of type A. Adding a vertex u of type A having one common neighbor with v produces a 4-critical graph G_2 (which is the result of applying the Hajos construction to two copies of K_4). If instead u has two common neighbors with v, checking cases shows that there is only one way to add a vertex w (of type B) to produce a 4-critical graph G_3.
Now suppose that we allow exactly one vertex v of type A. Checking cases shows that there is exactly one way to produce a 4-critical graph G_4, by adding two type B vertices, each having two consecutive vertices of the cycle as common neighbors with v.
Finally, suppose that we allow only vertices of type B. There are five possible placements of a type B vertex. Adding all five of them produces a 4-chromatic graph G_5, but deleting one produces a 3-chromatic graph. Thus G_5 must be 4-critical. □

Chromatic Number and 3-Decompositions

Theorem

There are exactly 29 fundamental 3-decompositions. These are given in the following table, where we let $C_{5,5} = C_5 + C_5$. (The generalized wheel, $W_{p,q} = C_p + K_q$, is $3+q$-critical if p is odd.)

$\{K_2, K_2, K_2\}$	$\{W_5, W_5, K_2\}$	$\{W_5, W_5, C_{5,5}\}$	$\{C_{5,5}, C_{5,5}, C_{5,5}\}$	
$\{W_5, K_3, G_1\}$	$\{W_5, K_3, G_2\}$	$\{W_5, K_4, G_3\}$	$\{W_5, K_4, G_4\}$	$\{W_5, K_6, G_5\}$
$\{C_{5,5}, W_{5,2}, G_1\}$	$\{C_{5,5}, W_{5,2}, G_2\}$	$\{C_{5,5}, W_{5,3}, G_3\}$	$\{C_{5,5}, W_{5,3}, G_4\}$	$\{C_{5,5}, W_{5,5}, G_5\}$
$\{G_1, G_1, K_4\}$				
$\{G_1, G_2, K_4\}$	$\{G_2, G_2, K_4\}$			
$\{G_1, G_3, K_5\}$	$\{G_2, G_3, K_5\}$	$\{G_3, G_3, K_6\}$		
$\{G_1, G_4, K_5\}$	$\{G_2, G_4, K_5\}$	$\{G_3, G_4, K_6\}$	$\{G_4, G_4, K_6\}$	
$\{G_1, G_5, K_7\}$	$\{G_2, G_5, K_7\}$	$\{G_3, G_5, K_8\}$	$\{G_4, G_5, K_8\}$	$\{G_5, G_5, K_{10}\}$

Proof.

By the lemma, there are exactly five extremal 2-decompositions that can appear in a fundamental 3-decomposition: $\{C_5, C_5\}$, $\{K_1, K_1\}$, $\{K_2, \overline{K}_2\}$, $\{K_3, \overline{K}_3\}$, and $\{K_5, \overline{K}_5\}$. Denote the first two as symmetric and the last three nonsymmetric. One of these five must be chosen for each of the three overlap sets of a fundamental 3-decomposition, but this choice is not independent. If a nonsymmetric 2-decomposition appears, then $\{C_5, C_5\}$ must also appear. Joining a pair of graphs from the 2-decompositions produces a color-critical graph except in the case $G = C_5 + \overline{K}_{n-5}$, $n \geq 7$, for which the lemma provides five possible color-critical subgraphs. □

Proof.

If all three 2-decompositions are symmetric, there are four possibilities, as given in the first row of the table.
Suppose exactly one nonsymmetric 2-decomposition appears. Then $\{C_5, C_5\}$ must also appear, and the third 2-decomposition can be either $\{C_5, C_5\}$ or $\{K_1, K_1\}$. Thus there are $5 \cdot 2 = 10$ possibilities, which are given in the second and third rows of the table.
Suppose exactly two nonsymmetric 2-decompositions appear, so $\{C_5, C_5\}$ is the third. Then we must choose two of the five color-critical subgraphs as factors, and the third must be a clique. Thus there are $\binom{5}{2} + 5 = 15$ possibilities, which are given in last five rows of the table. □

- We would like to determine all extremal 3-decompositions. Examples of some that are not fundamental include $\{K_p, K_p, C_{2p-1}\}$ or $\{K_p + C_5, K_p + C_5, C_{2p-1}\}$.
- Not all fundamental k-decompositions are produced by the algorithm for $k \geq 4$. Watkinson describes a decomposition of K_7 into $\{K_4, K_3, K_3, C_5\}$, though his presentation of this example contains an error. This example has a vertex contained in three critical subgraphs. The decomposition nonetheless has $\sum_{i=1}^{4} \chi(G_i) = n + \binom{4}{2} = 7 + 6 = 13$.

Chromatic Number and 3-Decompositions

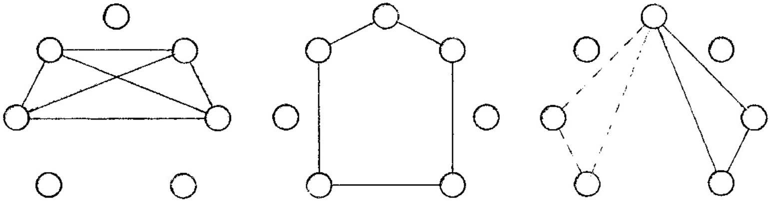


Figure: A 4-decomposition of K_7 into $\{K_4, K_3, K_3, C_5\}$ with a vertex contained in three factors.

- It appears unlikely that these results for $k = 3$ can be extended to list coloring. This is because there is no known formula expressing the list chromatic number of the join of two graphs in terms of the list chromatic numbers of the graphs.

Other Parameters

- We can determine analogous results for many other parameters. Furedi et al also considered the clique number.

Theorem

For all positive integers n and k with $n \geq \binom{k}{2}$, $\omega(k; K_n) = n + \binom{k}{2}$.

- The decomposition of the line graph $L(K_k)$ into k copies of K_{k-1} achieves $\omega(k; K_n) = n + \binom{k}{2}$.

Other Parameters

Theorem

We have $\alpha(k; K_n) = (k-1)n + 1$.

Proof.

Consider the decomposition $\{K_n, \overline{K}_n, ..., \overline{K}_n\}$. Then $\sum \alpha(G_i) = (k-1)n + 1$.
We use induction on order. Certainly $\alpha(k; K_1) = k$. Assume $\alpha(k; K_r) = (k-1)r + 1$, and let D be a decomposition of $G = K_{r+1}$. Consider the decomposition D' of $G - v$ formed by deleting v from each subgraph of D. If $\sum_{D'} \alpha(G_i) < (k-1)r + 1$, then $\sum_D \alpha(G_i) \leq (k-1)r + k = (k-1)(r+1) + 1$. If $\sum_{D'} \alpha(G_i) = (k-1)r + 1$, then by the pigeonhole principle, some vertex of K_r is contained in all k independent sets. Then v is contained in at most $k-1$ independent sets, so $\sum_D \alpha(G_i) \leq (k-1)r + 1 + (k-1) = (k-1)(r+1) + 1$. In either case, the result holds by induction. □

Other Parameters

- Note that the case $k=2$, $\alpha(2;K_n)=n+1$, is essentially the Nordhaus-Gaddum theorem due to the symmetry of complementation. The previous proposition and the decomposition it depends on also imply that the same formula holds for domination number and various related parameters.

Theorem

We have $\beta(k;K_n)=\left\lfloor \frac{n}{2}\right\rfloor \min\{k,\chi'(K_n)\}$.

Proof.

A decomposition containing $\min\{k,\chi'(K_n)\}$ copies of $\left\lfloor \frac{n}{2}\right\rfloor K_2$ shows that $\beta(k;K_n)\geq\left\lfloor \frac{n}{2}\right\rfloor \min\{k,\chi'(K_n)\}$. Equality must hold since $\beta(K_n)=\left\lfloor \frac{n}{2}\right\rfloor$ and there are at most $\chi'(K_n)$ such factors. □

Other Parameters

Theorem

We have $\triangle(k; K_n) = \binom{n}{2} - \binom{n-k}{2}$.

Proof.

Consider the decomposition with $G_i = K_{1,\max(n-i,0)}$ and any extra edges distributed arbitrarily. Then
$\sum \triangle(G_i) = \sum_{i=\max(n-k,0)}^{n-1} i = \binom{n}{2} - \binom{n-k}{2}$.
We use induction on order. If $k \geq n$, then
$\sum_D \triangle(G_i) \leq \sum_D m(G_i) = \binom{n}{2}$. If $k < n$, assume
$\triangle(k; K_r) = \binom{r}{2} - \binom{r-k}{2}$, and let D be a decomposition of
$G = K_{r+1}$. Let v be a vertex that does not uniquely have maximum degree in any of the k subgraphs. Consider the decomposition D' of $G - v$ formed by deleting v from each subgraph of D. Then adding v to the subgraphs of D' increases each maximum degree by at most one. Then $\sum_D \triangle(G_i) \leq \binom{r}{2} - \binom{r-k}{2} + k = \binom{r+1}{2} - \binom{r+1-k}{2}$.
The result holds by induction. □

Other Parameters

- Certainly $\chi'(k;K_n) \geq \triangle(k;K_n)$, and when $n-k \geq 2$ is even, $\chi'(k;K_n) \geq \triangle(k;K_n)+1$.

Conjecture

We have

$$\chi'(k;K_n) = \begin{cases} \Delta(k;K_n)+1 & n-k \geq 2\, even \\ \Delta(k;K_n) & else \end{cases}.$$

Thank You!

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