Nordhaus-Gaddum Theorems for k-Decompositions

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A Motivating Problem

Consider the following problem. An international round-robin sports tournament is held between n teams. The games are split between k locations in different countries, which can host multiple games simultaneously. The teams can travel to different locations to play, but it is impractical for the fans to visit more than one location. In this situation, it is reasonable to want teams that play at a given location to play as many games there as possible so that local fans can see them as much as possible. More precisely, we can compute the minimum number of games played by the teams at that location. We then wish to maximize the sum of these minimum numbers over all the locations in the tournament.

Definition

A graph G is composed of a set of vertices $V(G)$ and a set of edges $E(G)$, a subset of the set of 2-element subsets of $V(G)$.

Graphs are typically drawn with dots representing vertices and curves representing edges.

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A subgraph H of a graph G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

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Complete Graphs

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The complete graph K_n is the graph with order n and all possible edges. A complete subgraph of another graph is referred to as a clique.

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The degree of vertex v of graph G, $d_G(v)$, is the number of vertices that are adjacent to it. When the graph in question is clear, we write $d(v)$.

Definition

A graph parameter is a function $p(G)$ whose domain is all graphs. The minimum degree of a graph $\delta(G)$ is its smallest degree. The maximum degree of a graph $\Delta(G)$ is its largest degree. A graph G is regular if all vertices have the same degree (hence $\delta(G) = \Delta(G)$).

Definition

The chromatic number $\chi(G)$ is the smallest number of subsets into which the vertex set can be partitioned so that adjacent vertices are in distinct sets.

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Operations

Definition

A graph is p-critical with respect to parameter p if deleting any edge will reduce the value of the parameter. That is, $p(G-e) < p(G)$ for any edge e.

Definition

The join of G and H, $G+H$, has as its vertex set the disjoint union of the vertex sets of G and H, and its edge set contains all edges of G and H, and all edges between the copies of G and H.

Definition

The complement $\overline{G}$ of a graph G is a graph with the same vertex set and $E\left(\overline{G}\right) = \overline{E(G)}$ (the set complement relative to the set of all possible edges).

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Nordhaus-Gaddum Class Theorems

- One common way to study a graph parameter $p(G)$ is to examine the sum $p(G)+p(\overline{G})$ and product $p(G)\cdot p(\overline{G})$.
- Nordhaus and Gaddum proved the following Theorem for chromatic number in 1956.

Theorem

$$2\sqrt{n} \leq \chi(G)+\chi(\overline{G}) \leq n+1$$

$$n \leq \chi(G)\cdot\chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2$$

- Finck [1968] showed that all of these bounds are sharp and determined the extremal graphs.

Nordhaus-Gaddum Class Theorems

- Chartrand and Mitchem [1971] found similar bounds for other graph parameters.
- A theorem providing sharp upper and lower bounds for this sum and product is known as a theorem of the Nordhaus-Gaddum class.

Definition

A decomposition of a graph is a collection of subgraphs which partitions its edge set. A k-decomposition of a graph G is a decomposition of G into k subgraphs. The subgraphs in a decomposition are called factors.

- A graph and its complement decompose a complete graph. Hence a natural generalization of this problem is to consider decompositions into more than two factors.

Nordhaus-Gaddum Class Theorems

- The sum upper bound has attracted the most attention.

Definition

For a graph parameter p, let $p(k;G)$ denote the maximum of $\sum_{i=1}^{k} p(G_i)$ over all k-decompositions of G.

- Furedi, Kostochka, Stiebitz, Skrekovski, and West [2005] explored this upper bound for several different parameters.
- We will tend to describe a k-decomposition as $\{H_1, ..., H_k\}$, where each H_i is a p-critical subgraph.

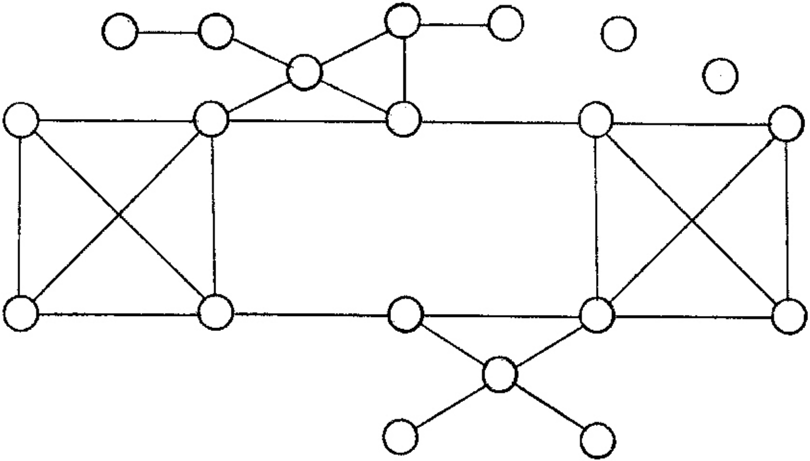
k-Cores

Definition

The k-core of a graph G, $C_k(G)$, is the maximal induced subgraph $H \subseteq G$ such that $\delta(H) \geq k$.
If D is the largest k such that G has a k-core, the D-core of G is called the maximum core.

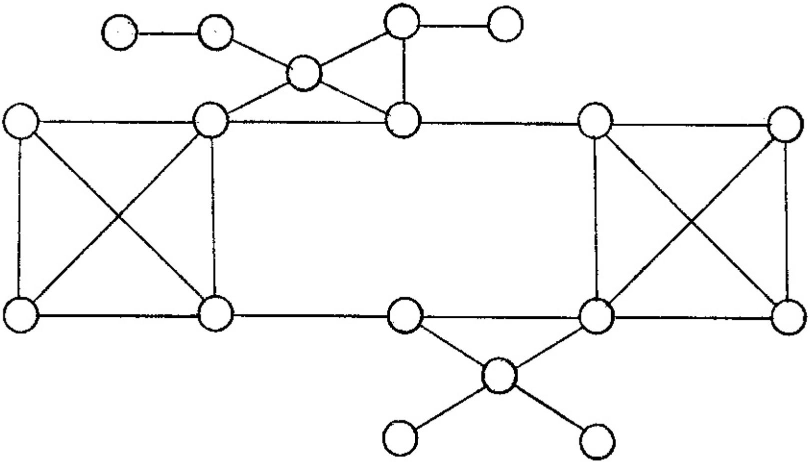
- The k-core was introduced by Steven B. Seidman in a 1983 paper entitled *Network structure and minimum degree*.
- It is immediate that the k-core is well-defined and that the cores are nested.
- It is immediate that the k-core of a graph must have order at least $k+1$.

k-Cores

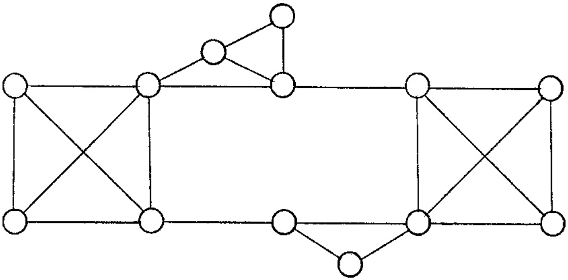


G is its own 0-core.

k-Cores



The 1-core of G.



The 2-core of G.

k-Cores

The 3-core of G is $2K_4$.

Definition

A vertex deletion sequence of a graph G is a sequence of vertices obtained by successively deleting a vertex of minimum degree from G. A vertex construction sequence of a graph G is obtained by reversing a vertex deletion sequence.
The degeneracy of G, $D(G)$, is the largest degree of a vertex when deleted by this process.

- It is immediate that $\delta(G) \leq D(G) \leq \triangle(G)$.

Definition

If the degeneracy and minimum degree of G are equal, $D(G) = \delta(G)$, we say G is k-monocore.

- Translating our motivating problem into graph theory terms, we wish to find $\delta(k; K_n)$ over all values of k and n. We will investigate $D(k; K_n)$, which may be the same thing.

k-Cores

- We need a way to determine the k-core of a graph. Iteratively deleting all vertices of degree less than k accomplishes this. We call this the k-core algorithm.

Theorem

Applying the k-core algorithm to graph G yields the k-core of G, provided it exists.

Theorem

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Degeneracy and 2-Decompositions

Theorem

a. We have $D(G) + D(\overline{G}) \leq n - 1$.
b. The graphs for which $D(G) + D(\overline{G}) = n - 1$ are exactly the graphs constructed by starting with a regular graph and iterating the following operation.
Given $k = D(G)$, H a k-monocore subgraph of G, add a vertex adjacent to at least $k + 1$ vertices of H, and all vertices of degree k in H (or similarly for $\overline{G}$).

Proof.

a. Let $p = D(G)$ and suppose $\overline{G}$ has an $n - p$-core. These cores use at least $(p + 1) + (n - p + 1) = n + 2$ vertices, and hence share a common vertex v. But then $d_G(v) + d_{\overline{G}}(v) \geq p + (n - p) = n$, a contradiction. □

Proof.

b. $(\Leftarrow)$ If G is regular with $k = D(G)$, then $\overline{G}$ is $n-k-1$-regular, so $D(G)+D\left(\overline{G}\right) = n-1$. If a vertex v is added as in the operation, producing a graph H, a $k+1$-core is produced, so $D(H)+D\left(\overline{H}\right) = (n+1)-1$.

$(\Rightarrow)$ Suppose that for a graph G, $D(G)+D\left(\overline{G}\right) = n-1$. If G and $\overline{G}$ are both monocore, then they must be regular. If G has a vertex v that is not contained in the maximum cores of both G and $\overline{G}$, then $D(G-v)+D\left(\overline{G}-v\right) = (n-1)-1$. Then v is contained in the maximum core of one of them, say G. Further, given $k = \widehat{C}(G)$, v is contained in a k-monocore subgraph H of G, and $H-v$ must be $k-1$-monocore. Then v must have been adjacent to all vertices of degree $k-1$ in $H-v$. Thus G can be constructed as described using the operation. $\square$

Degeneracy and 2-Decompositions

Theorem

[The degeneracy bound] For any graph G, $\chi(G) \leq 1 + D(G)$.

Proof.

Establish a construction sequence for G. Each vertex has degree at most $D(G)$ when colored. Coloring is uses at most one more color. Thus $\chi(G) \leq 1 + D(G)$. □

This upper bound is better than the more famous bound $\chi(G) \leq 1 + \Delta(G)$ (the maximum degree bound). This is a corollary, since $D(G) \leq \Delta(G)$.

Corollary

[Nordhaus-Gaddum] We have $\chi(G) + \chi(\overline{G}) \leq n + 1$.

Proof.

$\chi(G) + \chi(\overline{G}) \leq 1 + D(G) + 1 + D(\overline{G}) \leq n - 1 + 2 = n + 1.$ □

Degeneracy

- This theorem says that in any extremal 2-decomposition into spanning factors, they must be regular. This generalizes to k-decompositions.

Corollary

Let M be a k-decomposition of K_n into factors that are critical with respect to a maximum core number. Then $\sum_M (D(G_i)) \leq n-1$ with equality exactly for decompositions into k spanning regular graphs.

Proof.

Given vertex v, we have $\sum_M (D(G_i)) \leq \sum d_{G_i}(v) \leq n-1$. Equality holds exactly when every factor is regular. $\square$

- Next we consider k-decompositions with the restriction that each vertex is contained in exactly two factors. Consider the following construction.

Algorithm

Let r_1, ..., r_k be nonnegative integers at most one of which is odd. Let G_{ij}, $1 \leq i < j \leq k$ be an r_i-regular graph of order $r_i + r_j + 1$, and let $G_{ji} = \overline{G}_{ij}$. Let $G_i = \bigoplus_{j, j \neq i} G_{ji}$. Let S_k be the set of all k-decompositions of the form $\{G_1, ..., G_k\}$ constructed in this fashion.

The Two Factor Theorem

Theorem

[Two Factor Theorem] A k-decomposition with order $n > 1$ and every vertex in exactly two factors has $\sum_M (D(G_i)) \leq \left(\frac{2k-3}{k-1}\right) n - \frac{k}{2}$, and equality holds exactly for those decompositons in the set S_k.

Proof.

Since each vertex is contained in exactly two of the k factors, so we can partition them into $\binom{k}{2}$ distinct classes. Let $H_{ij} = V(G_i) \cap V(G_j)$ and let $n_{ij} = |H_{ij}|$ for $i \neq j$, $n_{ii} = 0$. Hence $n = \sum_{i,j} n_{ij}$. For $v \in H_{ij}$, we have $D(G_i) \leq d_{G_i}(v) \leq d_{G_i[H_{ij}]}(v) + \sum_{t=1}^{k} n_{it}$. Sum for each of the two factors and each of the $\binom{k}{2}$ classes. Then $(k-1)\sum_{i=1}^{k} D(G_i) \leq 2(k-2)\sum_{i,j} n_{ij} + \sum_{i,j,i\neq j}(n_{ij}-1) = (2k-3)n - \binom{k}{2}$, so $\sum_{i=1}^{k} D(G_i) \leq \left(\frac{2k-3}{k-1}\right) n - \frac{k}{2}$. □

Proof.

($\Rightarrow$) If this bound is an equality, then all k of the factors must be regular. Let $r_{ij} = d_{G_i[H_{ij}]}(v)$ for $v \in G_i[H_{ij}]$. Also, all edges between two classes sharing a common factor must be in that factor, so it is a join of $k-1$ graphs. A join of graphs is regular only when they are all regular. Now since G_i is regular, its complement must also be regular. But this implies that all the constants r_{ji}, $j \neq i$ are equal. Let r_i be this common value. Then $n_{ij} = r_i + r_j + 1$, so $n = (k-1)\sum r_i + \binom{k}{2}$. This implies that at most one of r_i and r_j is odd, so at most one of all the r_i's is odd.

($\Leftarrow$) Let G_i be a factor of a decomposition M constructed using the algorithm. It is easily seen that G_i is regular of degree $\left((k-3)r_i + (k-2) + \sum_j r_j\right)$. Summing over all the factors, we find that $\sum_M (D(G_i)) = \left(\frac{2k-3}{k-1}\right) n - \frac{k}{2}$. □

Degeneracy and 3-Decompositions

- Now consider 3-decompositions. The formula in the following theorem was proven by Furedi et al.

Theorem

We have $D(3;K_n) = \left\lfloor \frac{3}{2}(n-1) \right\rfloor$, and the extremal decompositions that avchieve $\sum_{i=1}^{3} D(G_i) = \frac{3}{2}(n-1)$ all consist of three $\frac{n-1}{2}$-regular graphs. For $n=1$, $\{K_1, K_1, K_1\}$ is the only extremal 3-decomposition, and for odd order $n>1$ they are exactly those in the set S_3.

Proof.

Let G_1, G_2, and G_3 be the three factors of an extremal decomposition for $D(3;K_n)$. It is obvious that $\{K_1, K_1, K_1\}$ is the only possibility for $n=1$, so let $n>1$. The previous theorem shows that $\sum_{i=1}^{3} D(G_i) \leq \frac{3}{2}(n-1)$. □

Proof.

Now any vertex can be contained in at most two of the three factors, since its degrees in the three graphs sum to at most $n-1$. Now adding a vertex with adjacencies so that it is contained in exactly one of the three factors increases n by one and $\sum_{i=1}^{3} D(G_i)$ by at most one, so this cannot violate the bound. Thus deleting a vertex of an extremal decomposition contained in only one of the three factors would decrease n by one and $\sum_{i=1}^{3} D(G_i)$ by at most one. For n odd, this is a contradiction and for n even it can occur only when it is the only such vertex.

If there are only two distinct classes, then add a vertex joined to all the vertices of the two disjoint factors. This increases n by one and $\sum_{i=1}^{3} D(G_i)$ by two. Hence if the new decomposition satisfies the bound, so does the original, and if the orginal decomposition attains the bound, then n must be even. $\square$

Proof.

Thus by the Two Factor Theorem, those decompositions with $\sum_{i=1}^{3} D(G_i) = \frac{3}{2}(n-1)$ are exactly those in S_3. Further, by the proof of this theorem the factors of such a decomposition are all $1+\sum r_j$-regular. Now $2(r_1+r_2+r_3) = \sum(n_{ij}-1) = n-3$, so $\sum_j r_j = \frac{n-3}{2}$. Thus the factors are all $\frac{n-1}{2}$-regular.
Finally note that joining a vertex to all vertices of one factor of an extremal decomposition of odd order attains the bound for even order, so $D(3;K_n) = \left\lfloor \frac{3}{2}(n-1) \right\rfloor$ for even orders as well. □

- An extremal decomposition of even order can be formed from one of odd order by either joining a vertex to one of the factors or deleting a vertex contained in two factors. However, the decomposition $\{K_4, C_4, C_4\}$ shows that not all extremal decompositions of even order can be formed this way.

Degeneracy and 4-Decompositions

- Furedi et al also proved that $D(4;K_n) = \left\lfloor \frac{5}{3}(n-1) \right\rfloor$. We use their proof to show that all extremal decompositions with $n = 3r+1 > 1$ can be constructed by the following algorithm.

Algorithm

Let n, r, a, b, c, and s be nonnegative integers with $n = 3r+1$, $a+b+c = s-1$ and a, b, c, even if s is odd. Let G_1, G_2, G_3 be a, b, c-regular graphs, respectively, of order s. Let G_4, G_5, G_6 be $r-s$-regular graphs of orders $r-a$, $r-b$, $r-c$, respectively. Let S be the set of all decompositions of the form
$\left\{G_1+G_4, G_2+G_5, G_3+G_6, \overline{G}_3+\overline{G}_4+\overline{G}_6\right\}$.

Theorem

We have $D(4;K_n) = \left\lfloor \frac{5}{3}(n-1) \right\rfloor$. For $n=1$, $\{K_1,K_1,K_1,K_1\}$ is the only extremal 4-decomposition and the extremal decompositions of order $n = 3r+1 > 1$ that achieve $\sum_{i=1}^{4} D(G_i) = \frac{5}{3}(n-1)$ are exactly those in S.

Proof.

It is obvious that $\{K_1, K_1, K_1, K_1\}$ is the only possibility for $n = 1$, so let $n > 1$. It is easily checked that the decompositions in S exist and achieve the stated sum. Joining a vertex to one of the factors achieves the stated bound for $n = 3r + 2$, and deleting a vertex contained in two of the factors achieves the bound for $n = 3r$.
As in the previous theorem, it is easily shown that no vertex is contained in a single factor or all four factors. If each vertex is contained in exactly two of the four factors, then the Two Factor Theorem says that $\sum_{i=1}^{4} D(G_i) \leq \frac{5}{3}n - 2$. Hence this decomposition is not extremal for $n = 3r + 1$. $\square$

Proof.

Consider an extremal decomposition with a vertex contained in three of the factors. Call these factors 1, 2, and, 3 so that $D(G_1) \le D(G_2) \le D(G_3)$. Let $H_{123} = V(G_1) \cap V(G_2) \cap V(G_3)$ and $H_{i4} = V(G_i) \cap V(G_4)$. Then $D(G_1) + D(G_2) + D(G_3) \le n-1$, so $D(G_1) + D(G_2) \le \frac{2}{3}(n-1)$. Now $D(G_3) + D(G_4) \le n-1$, so $\sum_{i=1}^{4} D(G_i) \le \frac{5}{3}(n-1)$, and $D(4;K_n) = \left\lfloor \frac{5}{3}(n-1) \right\rfloor$.
If $\sum_{i=1}^{4} D(G_i) = \frac{5}{3}(n-1)$, then $D(G_1) + D(G_2) = \frac{2}{3}(n-1)$ and $D(G_3) + D(G_4) = n-1$. The former implies that $D(G_1) = D(G_2) = D(G_3) = \frac{1}{3}(n-1)$. The latter and this imply that $D(G_4) = \frac{2}{3}(n-1)$ and each vertex in $G_i \cap G_4$, $i \in \{1,2,3\}$, is only adjacent to vertices in these two factors. Hence the vertices partition into H_{123} and $H_{i4} = G_i \cap G_4$, $i \in \{1,2,3\}$ whose orders we call n_{123} and n_{i4}, respectively. Furthermore, each of the factors is regular. $\square$

Proof.

Then H_{123} is decomposed into three regular spanning factors whose degrees are even if n_{123} is odd, and the other sets are decomposed into two regular spanning graphs. Let $r_{i,S} = d_{G_i[H_S]}(v)$ for $v \in G_i[H_S]$. Hence $r_{1,123} + n_{14} = r_{1,14} + n_{123}$, $r_{2,123} + n_{24} = r_{2,24} + n_{123}$, and $r_{3,123} + n_{34} = r_{3,34} + n_{123}$. Now since G_4 is regular, so is $\overline{G}_4$. Thus $r_{1,14} = r_{2,24} = r_{3,34}$, so each of the factors is regular of the same degree. Let $r = \frac{1}{3}(n-1)$ be this common value, $s = n_{123}$, so $r - s = r_{1,14} = r_{2,24} = r_{3,34}$. Let $a = r_{1,123}$, $b = r_{2,123}$, and $c = r_{3,123}$, so $a + b + c = s - 1$, $n_{14} = r - a$, $n_{24} = r - b$, and $n_{34} = r - c$. There are no parity problems, so the extremal decomposition can be constructed by the algorithm. $\square$

Bounds for k=5 and 6

- The values of $D(k;K_n)$ for $k \in \{2,3,4\}$ satisfy $D(k;K_n) = \left\lfloor \frac{2k-3}{k-1}(n-1) \right\rfloor$. In fact, Furedi et al produced a simple construction to prove that $D(k;K_n) \geq \left\lfloor \frac{2k-3}{k-1}(n-1) \right\rfloor$, but this is not an equality for $k \geq 5$.

Algorithm

Let S_5^* be the set of all decompositions that can be constructed as follows. Take a decomposition M in S_4 with the additional property that the sum of some two of the four r_i's equals the sum of the other two r_i's (e.g. $r_1 + r_2 = r_3 + r_4$). Let r be this common value. Add the factor $K_{r+1,r+1}$ to the decomposition.

Bounds for k=5 and 6

Theorem

We have $D(5;K_n) \geq \left\lfloor \frac{11}{6}n - 2 \right\rfloor$.

Proof.

This construction has $\sum_{i=1}^{5} D(G_i) = \frac{5}{3}n - 2 + \frac{n}{6} = \frac{11}{6}n - 2$ for any order that it can attain. The proof of the Two Factor Theorem shows that a decomposition in S_k has order $n = (k-1)\sum r_i + \binom{k}{2}$. For $k = 4$, this gives $n = 3\sum r_i + 6$. To satisfy the property in the construction, all the r_i's must be even, and it is obvious that any nonnegative even r can be attained. Hence for each positive order $n = 6r$ there is a decomposition in S_5^* with this order. Successively deleting five vertices contained in exactly two factors from such a decomposition provides decompositions attaining the bound for the other five classes of orders mod 6. $\square$

Bounds for k=5 and 6

Conjecture

For $n \geq 2$, $D(5; K_n) = \left\lfloor \frac{11}{6} n - 2 \right\rfloor$.

- The best known upper bound, due to Furedi et al says that $D(5; K_n) \leq 2n - 3$.

Algorithm

Let S_6^* be the set of all decompositions that can be constructed as follows. Take a decomposition M in S_4 with the additional property that two pairs of two of the four r_i's are equal. (e.g. $r_1 = r_2$ and $r_3 = r_4$). Let r be the sum of these two values. Add two copies of the factor $K_{r+1,r+1}$ to the decomposition.

Theorem

For $n \geq 4$, $D(6; K_n) \geq 2n - 2$.

Bounds for k=5 and 6

Proof.

This construction has $\sum_{i=1}^{6} D(G_i) = \frac{5}{3}n - 2 + 2\left(\frac{n}{6}\right) = 2n - 2$ for any order that it can attain. The proof of the Two Factor Theorem shows that a decomposition in S_k has order $n = (k-1)\sum r_i + \binom{k}{2}$. For $k = 4$, this gives $n = 3\sum r_i + 6$. To satisfy the property in the construction, all the r_i's must be even, and any nonnegative even $r = 4s$ can be attained. Hence for each positive order $n = 12s + 6$ there is a decomposition in S_6^* with this order. Successively deleting vertices contained in exactly two factors from such a decomposition provides decompositions attaining the bound when $4 \le n \le 6$, $12 \le n \le 18$, and $n \ge 20$. Joining a vertex to each of the two disjoint factors when $n = 12s + 6$ works for $n \in \{7, 19\}$. Now $\{2[K_4], 4[C_4]\}$ works for $n = 8$ and $\{3[K_4], 3[K_{3,3}]\}$ works for $n = 10$. Joining a vertex to disjoint factors in these last two works for $n \in \{9, 11\}$. $\square$

Bounds for k=5 and 6

Conjecture

For $n \geq 4$, $D(6; K_n) = 2n - 2$.

- The best known upper bound, due to Furedi et al says that $D(6; K_n) \leq \frac{5}{2}n - \frac{7}{2}$.
- The constructions that we have seen so far start with a small decomposition and 'expand' it to a bigger one. In some cases, this process can be generalized.

Expanded Constructions

Theorem

Suppose there is a k-decomposition of K_n into regular subgraphs and $\sum_{i=1}^{k} D(G_i) = c(n-1)$. Then there are infinitely many other k-decompositions with order n' and $\sum_{i=1}^{k} D(G_i) = c(n'-1)$.

Proof.

Let $r = n-1$. Let M be a decomposition of K_{rt+1} into r t-regular spanning factors, where t is even if r is even. Form a k-decomposition M' with order n' by replacing each vertex of K_n with a copy of M so that if vertex v has degree d_i in G_i, then d_i of the r factors are merged together. Finally, join the corresponding factors in different copies of M.

If the factor G_i has degree d_i in K_n, then the factor G_i' has degree $d_i(rt+1) + d_i t$. Now since $\sum_{i=1}^{k} d_i = c(n-1)$ and $n' = n(rt+1)$,
$\sum_{i=1}^{k} (d_i(rt+1) + d_i t) = (rt+1+t)\sum d_i = (rt+1+t)c(n-1)$
$= c[n(rt+1) - 1 + t(n-1-r)] = c(n'-1)$. □

Expanded Constructions

- We now consider a number of decompositions that can be expanded to infinite families via the previous theorem.
- Decompose K_n into $k = \binom{n}{2}$ K_2's. Then $\sum_{i=1}^{k} D(G_i) = \binom{n}{2} = \frac{n}{2}(n-1) = \frac{1+\sqrt{1+8k}}{4}(n-1)$. Thus this sum can be achieved for infinitely many orders whenever k is a triangular number.
- Decompose K_n into K_3's, which can occur whenever $n \equiv 1$ or 3 mod 6. Such a decomposition has $k = \frac{1}{3}\binom{n}{2} = \frac{n(n-1)}{6}$ triangles, so $\sum_{i=1}^{k} D(G_i) = 2\frac{n(n-1)}{6} = \frac{n}{3}(n-1) = \frac{1+\sqrt{1+24k}}{6}(n-1)$.
- In particular, consider $k = 7$. Let H be an r-regular graph of order $3r+1$. Let $G = H + H + H$. Then G is $7r+2$-regular, and 7 copies of G form a decomposition of order $n = 7(3r+1) = 21r+7$, so $\frac{n-1}{3} = 7r+2$. Then $\sum_{i=1}^{7} D(G_i) = 7(7r+2) = \frac{7}{3}(n-1)$. This construction shows that $D(7;K_n) \geq \left\lfloor \frac{7}{3}(n-1) \right\rfloor$ for $n = 7(3r+1)$.

Expanded Constructions

- Decompose K_n into K_4's, which can occur whenever $n \equiv 1$ or 4 mod 12 [Hanani 1961]. Such a decomposition has $k = \frac{1}{6}\binom{n}{2} = \frac{n(n-1)}{12}$ K_4's, so $\sum_{i=1}^{k} D(G_i) = 3\frac{n(n-1)}{12} = \frac{n}{4}(n-1) = \frac{1+\sqrt{1+48k}}{8}(n-1)$.
- Decompose K_n into K_5's, which can occur whenever $n \equiv 1$ or 5 mod 20 [Hanani 1975]. Such a decomposition has $k = \frac{1}{10}\binom{n}{2} = \frac{n(n-1)}{20}$ K_5's, so $\sum_{i=1}^{k} D(G_i) = 4\frac{n(n-1)}{20} = \frac{n}{5}(n-1) = \frac{1+\sqrt{1+80k}}{10}(n-1)$.
- Let $n = p^2 + p + 1$, where p is a prime power. Then there is a projective plane with n points and n lines, which correspond to vertices and factors of a decomposition. Then $\sum_{i=1}^{k} D(G_i) = kp = \frac{kp}{k-1}(n-1) = \frac{p^2+p+1}{p+1}(n-1) = \frac{\left(-1+\sqrt{4k-3}\right)k}{2(k-1)}(n-1)$.
- Let $k[G]$ mean that factor G occurs k times in a decomposition.

Expanded Constructions

k	$\sum D(G_i)$	decomposition
2	$n-1$	$\{2[K_1]\}$
3	$\frac{3}{2}(n-1)$	$\{3[K_2]\}$
4	$\frac{5}{3}(n-1)$	$\{K_3, 3[K_2]\}$
5	$\frac{9}{5}(n-1)$	$\{4[K_3], 3K_2\}$
6	$2(n-1)$	$\{6[K_2]\}$
7	$\frac{7}{3}(n-1)$	$\{7[K_3]\}$
8	$\frac{9}{4}(n-1)$	$\{K_3, 7[K_2]\}$
9	$\frac{12}{5}(n-1)$	$\{3[K_3], 6[K_2]\}$
10	$\frac{5}{2}(n-1)$	$\{10[K_2]\}$
11	$\frac{19}{7}(n-1)$	$\{8[K_3], 2[K_2], 2K_2\}$
12	$3(n-1)$	$\{12[K_3]\}$
13	$\frac{13}{4}(n-1)$	$\{13[K_4]\}$
14	$\frac{25}{8}(n-1)$	$\{11[K_3], 3[K_2]\}$
15	$3(n-1)$	$\{15[K_2]\}$
16	$\frac{17}{5}(n-1)$	$\{K_5, 15[K_3]\}$
20	$4(n-1)$	$\{20[K$[illegible]$]\}$

- There is another way to generate decompositions that are better for some orders. If a decomposition has $\sum_{i=1}^{k} D(G_i) = c(n-1)$, then some factor G_i has $D(G_i) \leq \frac{c}{k}(n-1)$. Generalizing this, we have the following.

Theorem

If there is a decomposition of K_n with $\sum_{i=1}^{k} D(G_i) = c(n-1)$, then given $0 \leq p \leq k-1$, there is a decomposition of K_n with $\sum_{i=1}^{k-p} D(G_i) \geq c\frac{k-p}{k}(n-1)$.

- Furedi et al also proved the general upper bound that for all positive integers n and k, $D(k;K_n) \leq \sqrt{k} \cdot n$. This is not attained for any values of n and k. Using essentially the same approach, this can be strengthened to a sharp bound.

A General Upper Bound

Theorem

For all positive integers n and k, we have $D(k;K_n) \le -\frac{k}{2} + \sqrt{\frac{k^2}{4} + kn(n-1)}$. This is an equality exactly when there is a decomposition of K_n into k cliques of equal size.

Proof.

For a k-decomposition, let $d_i = D(G_i)$ and $F = \sum \widehat{C}(G_i)$. Then $m(G_i) \ge \binom{d_i+1}{2}$. Now

$$\frac{n(n-1)}{2} = \binom{n}{2} \ge \sum_{i=1}^{k} \binom{d_i+1}{2} = \frac{1}{2}\sum_{i=1}^{k}(d_i^2 + d_i) \ge \frac{1}{2}\left(\frac{F^2}{k} + F\right).$$

The first inequality is attained exactly when all the factors are cliques, and the second is attained exactly when all the cliques have the same size. Hence $kn(n-1) \ge F^2 + kF$, so $F^2 + kF - kn(n-1) \le 0$, and $F \le -\frac{k}{2} + \sqrt{\frac{k^2}{4} + kn(n-1)}$. $\square$

A General Upper Bound

- We can obtain the successively simpler but weaker formulas $D(k;K_n) \leq -\frac{k}{2}+\sqrt{\frac{k^2}{4}+kn(n-1)} < \sqrt{kn(n-1)} < \sqrt{k}\left(n-\frac{1}{2}\right) < \sqrt{k}\cdot n$ as corollaries. The last is the bound reported by Furedi et al.
- A decomposition of K_n into k cliques of equal size is a block design. In particular, it is a $\left(n, k, \frac{k+\sqrt{k^2+4kn(n-1)}}{2n}, \frac{1}{2}+\sqrt{\frac{1}{4}+\frac{n(n-1)}{k}}, 1\right)$-design. Hence the previous result will attain equality whenever such a design exists.

A General Upper Bound

Corollary

We have

1. $D\left(\binom{n}{2}; K_n\right) = \binom{n}{2}$ *for* $n \geq 2$
2. $D\left(\frac{n(n-1)}{6}; K_n\right) = \frac{n(n-1)}{3}$ *for* $n \equiv 1$ *or 3 mod 6*
3. $D\left(\frac{n(n-1)}{12}; K_n\right) = \frac{n(n-1)}{4}$ *for* $n \equiv 1$ *or 4 mod 12*
4. $D\left(\frac{n(n-1)}{20}; K_n\right) = \frac{n(n-1)}{5}$ *for* $n \equiv 1$ *or 5 mod 20*
5. $D(n; K_n) = \frac{\left(-1+\sqrt{4n-3}\right)n}{2}$ *for* $n = p^2 + p + 1$*, where* p *is a prime power.*

Thank you!

For more information, see

A. Bickle, Nordhaus-Gaddum Theorems for k-Decompositions, Congr. Num. 211 (2012) 171-183.

Z. Furedi, A. Kostochka, M. Stiebitz, R. Skrekovski, and D. West, Nordhaus-Gaddum-type theorems for decompositions into many parts. J. Graph Theory 50 (2005), 273-292.