Graphs Classes With Size $kn - \binom{k+1}{2}$

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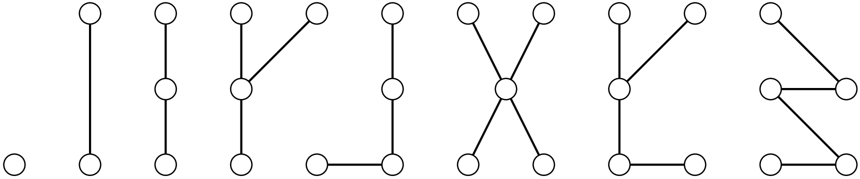
Definition

A **graph** G has a set of **vertices** and a set of **edges.** An edge is two-element subset of the vertex set.
A **graph class** is a set of graphs.
The **order** $n(G)$ of a graph G is the number of vertices of G. The **size** $m(G)$ of a graph G is the number of edges of G.

- Many graph classes have sizes that are determined, or bounded by, their orders. Some formulas occur repeatedly.
- The k-**tree size** is $m(G) = k \cdot n(G) - \binom{k+1}{2}$.
- $n-1$, $2n-3$, $3n-6$, $4n-10$, ...
- $n-1$: trees
- $2n-3$: outerplanar graphs
- $3n-6$: planar graphs

Definition

A **tree** is a connected graph that does not contain a cycle.



Proposition

Every nontrivial tree contains at least two leaves.

Proof.

If not, a maximal path could be extended to form a cycle. □

Theorem

A graph is a tree if and only if it can be constructed from K_1 by repeatedly applying the operation of adding a new vertex and making it adjacent to one existing vertex.

Proof.

($\Leftarrow$) K_1 is a tree, and the operation keeps the graph connected and acyclic. Thus any graph produced this way must be a tree.
($\Rightarrow$) We use induction on n. The result is obvious when $n = 1$. Assume that any tree of order $n - 1$ can be constructed from K_1 using the operation, and let T have order $n > 1$. Then T has a leaf v, so $T - v$ is a tree with order $n - 1$. Thus $T - v$ can be constructed using the operation, so T can also. $\square$

Trees

Theorem

A graph is a tree if and only if it can be constructed from K_1 by repeatedly applying the operation of adding a new vertex and making it adjacent to one existing vertex.

Corollary

A tree with order n has size $n-1$.

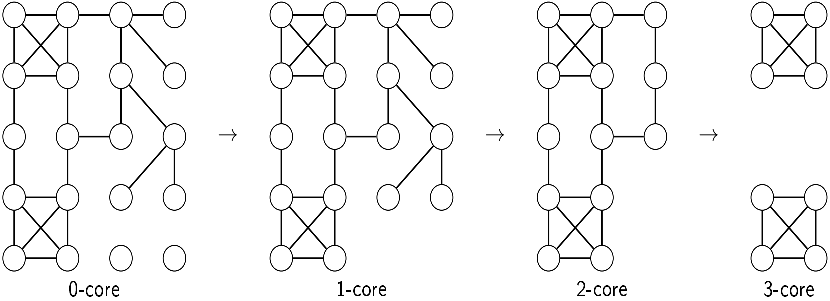
Proof.

The operation adds an edge for each new vertex, and $m(K_1) = 0$. $\square$

- There are several ways of generalizing trees. We generalize the operation of adding one new vertex adjacent to one other vertex.

Definition

The k-**core of a graph** G, is the maximal induced subgraph $H \subseteq G$ such that $\delta(G) \geq k$, if it exists. A graph is k-**core-free** if it does not contain a k-core. The **core number** of a vertex, $C(v)$, is the largest value for k such that $v \in C_k(G)$.



Cores

- Cores were introduced by S. B. Seidman [1983] and have been studied extensively in (Bickle [2010,2013]).
- Seidman briefly explores applications to social networks in his paper.
- Cores also have applications in computer science to network visualization (ADBV [2006], Gaertler/Patrignani [2004]).
- They also have applications to bioinformatics (ANKMSAWM [2003], Bader/Hogue [2003], Wuchty/Almaas [2005]).
- There is a simple algorithm for determining the k-core of a graph, which we shall call the k-Core Algorithm.

Algorithm

(k-**Core Algorithm**) *Input a graph G and integer k. Iterate the step of deleting all vertices of degree less than k. Stop when there are no more such vertices. If a graph remains, it is the k-core. If no graph remains, G has no k-core.*

Cores

Theorem

Applying the k-Core Algorithm to graph G yields the k-core of G, if it exists. That is, a vertex v is in the k-core of G if and only if it is not deleted by the algorithm.

Proof.

($\Rightarrow$) The vertices in the k-core all have at least k neighbors in the k-core. None of these vertices will be deleted in the first iteration. If none have been deleted after i iterations, none will be deleted by the next iteration. Thus none will ever be wrongly deleted.
($\Leftarrow$) The vertices not deleted by the algorithm all have degree at least k in the graph produced by the algorithm. Thus they are all in the k-core, so no vertices will be wrongly included. $\square$

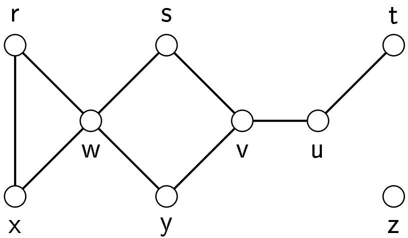
Degeneracy

- The k-Core Algorithm can be implemented in polynomial time. ($O\left(n^2\right)$ time using an adjacency matrix, or $O(m)$ time using an edge list, which is better for sparse graphs).
- We can define a sequence of vertices based on the order that they are deleted by the k-Core Algorithm. We may also wish to construct a graph by successively adding vertices of relatively small degree.

Definition

A **deletion sequence** of a graph G is a sequence of its vertices formed by iterating the operation of deleting a vertex of smallest degree and adding it to the sequence until no vertices remain. A **construction sequence** of a graph is the reversal of a corresponding deletion sequence. A graph is k-**degenerate** if its vertices can be successively deleted so that when deleted, each has degree at most k. The **degeneracy** $D(G)$ of a graph G is the smallest k such that it is k-degenerate.

Example. A deletion sequence of the graph below is $z, t, u, s, v, y, r, w, x$. The graph has degeneracy 2.



- The term k-degenerate was introduced in 1970 by Lick and White; the concept has been introduced under other names both before and since. As a corollary of Theorem 9, we have the following min-max relationship.

Corollary

For any graph, its maximum core number is equal to its degeneracy.

Proof.

Let G be a graph with degeneracy D and maximum core number k. By Theorem 9, since G has a k-core, it is not $(k-1)$-degenerate, so $k \leq D$. Since G has no $k+1$-core, it is k-degenerate, so $k = D$. □

- The size of maximal k-degenerate graphs is a basic result.

Degeneracy

Theorem

The size of a maximal k-degenerate with order $n \geq k$ is $k \cdot n - \binom{k+1}{2}$.

Proof.

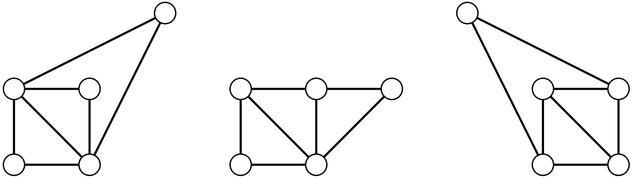
If G is k-degenerate, then its vertices can be successively deleted so that when deleted they have degree at most k. Since G is maximal, the degrees of the deleted vertices will be exactly k until the number of vertices remaining is at most k. After that, the $n-j^{th}$ vertex deleted will have degree j. Thus the size m of G is

$$\begin{aligned}
m &= \sum_{i=0}^{k-1} i + \sum_{i=k}^{n-1} k \\
&= \frac{k(k-1)}{2} + k(n-k) \\
&= k \cdot n + \frac{k(k-1)}{2} - \frac{2k^2}{2} \\
&= k \cdot n - \binom{k+1}{2}.
\end{aligned}$$

Degeneracy

- Thus a k-degenerate graph is maximal if and only if it has size $k \cdot n - \binom{k+1}{2}$.

Example. The three maximal 2-degenerate graphs of order 5 are shown below.



Corollary

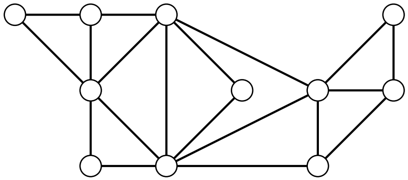
Every graph with order n and size $m \geq (k-1)n - \binom{k}{2} + 1$, $1 \leq k \leq n-1$, has a k-core.

- One class of maximal k-degenerate graphs is particularly important.

Definition

A k-**tree** is a graph that can be formed by starting with K_{k+1} and iterating the operation of making a new vertex adjacent to all the vertices of a k-clique of the existing graph. The clique used to start the construction is called the **root** of the k-tree.

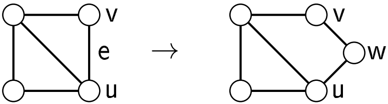
The graph below is a 2-tree. Any triangle could be the root.



- tree = 1-tree = maximal 1-degenerate
- k-trees $\subseteq$ maximal k-degenerate graphs

Definition

A **subdivision of an edge** $e = uv$, deletes e and adds vertex w and edges uw and wv. A graph H is a **subdivision of a graph** G if it can be obtained by some number (perhaps zero) of subdivisions of edges of G.

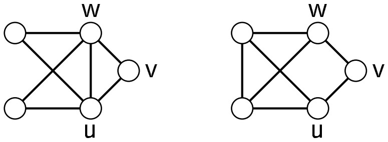


Theorem

(Bickle [2012]) *A maximal k-degenerate graph is a k-tree if and only if it contains no subdivision of* K_{k+2}.

Proof.

($\Rightarrow$) Let G be a k-tree. Certainly K_{k+1} contains no subdivision of K_{k+2}. Suppose G is a counterexample of minimum order with a vertex v of degree k. Then $G-v$ is a k-tree with no subdivision of K_{k+2}, so the subdivision in G contains v. But then v is not one of the $k+2$ vertices of degree $k+1$ in the subdivision, so it is on a path P between two such vertices. Let its neighbors on P be u and w. But since the neighbors of v form a clique, $uw \in G-v$, so P could avoid v, implying $G-v$ has a subdivision of K_{k+2}. This is a contradiction. $\square$



Theorem

(Bickle [2012]) *A maximal k-degenerate graph is a k-tree if and only if it contains no subdivision of* K_{k+2}.

Proof.

($\Leftarrow$) (contrapositive) Let G be maximal k-degenerate and not a k-tree. Since G is constructed beginning with a k-tree, for a given construction sequence there is a first vertex in the sequence that makes G not a k-tree. Let v be this vertex, and H be the maximal k-degenerate subgraph induced by the vertices of the construction sequence up to v. Then $n(H) \geq k+3$, $d_H(v) = k$, v has nonadjacent neighbors u and w, and $H - v$ is a k-tree. Now there is a sequence of at least two $k+1$-cliques starting with one containing u and ending with one containing w, such that each pair of consecutive $k+1$-cliques in the sequence overlap on a k-clique. Then two of these cliques and a path through v produces a subdivision of K_{k+2}.

Definition

A **plane drawing** of a graph is a drawing in the plane that has no crossings. A graph is **planar** if it has a plane drawing. A **nonplanar** graph is not planar. The **regions** of a plane drawing are the maximal pieces of the plane surrounded by edges and vertices. The infinite region is the **exterior region**. The **boundary** of a region is the subgraph induced by the edges that touch it. The **length** of a region is the length of a walk around it.

- There is a basic relationship between the number of vertices, edges, and regions of a planar graph.

Theorem

(**Euler's Polyhedron Formula**) *For a connected planar graph with order n, size m, and r regions, $n - m + r = 2$.*

Theorem

(**Euler's Polyhedron Formula**) *For a connected planar graph with order n, size m, and r regions, $n - m + r = 2$.*

Proof.

We use induction on m. For a connected graph G, m is minimum when $m = n - 1$, and G is a tree. Then $r = 1$, so $n - (n-1) + 1 = 2$, as desired.
Assume the formula holds for graphs with size less than m, and let G be a connected planar graph with size $m > n - 1$. Then G contains a cycle. Let e be an edge of a cycle. Now $G - e$ is connected and planar with size $m - 1$. It has $r - 1$ regions, since the two regions bordering e merge into one when e is deleted. Then $n - m + r = n - (m-1) + (r-1) = 2$, so the formula holds for G. $\square$

Planarity

Theorem

The size of a planar graph with $n \geq 3$ satisfies $m \leq 3n-6$.

Proof.

Let G be a planar graph with order n, size m, and r regions with lengths r_i. Each region uses at least three edges, and each edge is used twice in region boundaries, so $3r \leq \sum r_i = 2m$. Now $n-m+r=2$, so $6 = 3n-3m+3r \leq 3n-3m+2m = 3n-m$. Thus $m \leq 3n-6$. □

Definition

A graph is **maximal planar** if no edge can be added without making it nonplanar. A plane drawing of a graph is a **triangulation** if every region is a triangle.

Corollary

The following are equivalent for a planar graph G.

1. G is maximal planar.

2. G has $m = 3n - 6$.

3. A plane drawing of G is a triangulation.

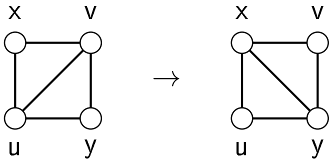
Proof.

$(1 \Leftrightarrow 3)$ A plane drawing of G is a triangulation if and only if there is no region with length longer than 3, since otherwise a chord could be added.

$(2 \Leftrightarrow 3)$ Following the proof of Theorem 21, every region is a triangle if and only if $3r = 2m$ if and only if $m = 3n - 6$. □

Planarity

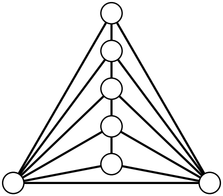
- A maximal planar graph may be a 3-tree. Indeed, any maximal planar graph formed by starting with K_3 and adding a vertex adjacent to three vertices of a region is a 3-tree. But maximal planar graphs need not be 3-trees, or even 3-degenerate. Any planar graph is 5-degenerate, and there are planar graphs with degeneracy 4 and 5.
- Let G be a maximal planar graph with edge $e = uv$, and x and y the other vertices of the triangular regions containing e. A **flip** of uv deletes uv and adds xy (assuming it is not already an edge).



Planarity

Theorem

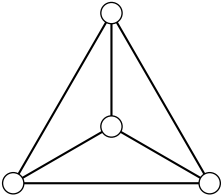
(Wagner [1936]) *Any maximal planar graph can be converted to (the 3-tree) $P_{n-2} + K_2$ via a sequence of flips.*



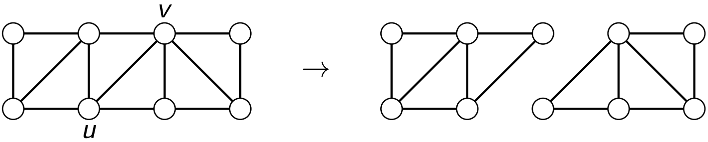
- Sketch: Start with a vertex with maximum degree and flip edges until it is adjacent to all other vertices. Repeat the process with a second vertex. The resulting graph must be $P_{n-2} + K_2$.

Definition

A graph is **outerplanar** if it has a plane drawing with all vertices on the exterior region.



The graphs K_4 and $K_{2,3}$ are not outerplanar, since any drawing of them must have one vertex in the interior.



Theorem

Any outerplanar graph is 2-degenerate. In particular, any nontrivial outerplanar graph has at least two vertices of degree at most two.

Proof.

The result is obvious when $2 \leq n \leq 4$, where K_4 is the only non-outerplanar graph. Assume the result holds for all maximal outerplanar graphs with order less than n, and let G be a maximal outerplanar graph of order n. Then all the vertices are on a (Hamiltonian) cycle C. Every other edge uv is a chord of C. The vertices of the two $u-v$ paths on C induce two maximal outerplanar graphs than only overlap on $\{u, v\}$. By induction, each of them has at least two vertices of degree at most two, at least one of which is not u or v. Thus the result holds for G. Thus it holds for any nontrivial outerplanar graph. Thus all outerplanar graphs are 2-degenerate. $\square$

K_4 Subdivisions

Corollary

The size of a nontrivial outerplanar graph satisfies $m \leq 2n - 3$.

- A maximal outerplanar graph, which is a 2-tree, has $m = 2n - 3$.
- Any subdivision of K_3 is a cycle, so the maximum size of a graph with no K_3 subdivision is $n - 1$.
- What guarantees the existence of a subdivision of K_4?

Definition

An S-**lobe** of a graph G is a subgraph of G induced by a cutset S and a component of $G - S$.

Lemma

(Dirac [1960]) *If G is a k-connected graph and v, v_1, ..., v_k are distinct vertices of G, then there are independent $v - v_i$ paths for $1 \leq i \leq k$.*

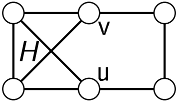
K_4 Subdivisions

Theorem

(Dirac [1964]) *Every graph with at most one vertex with degree less than 3 contains a subdivision of* K_4.

Proof.

We use induction on order n. The smallest order with a graph (K_4) satisfying the hypothesis is $n = 4$, which certainly satisfies the conclusion. Assume the result holds for all graphs with order n', $4 \leq n' < n$, and G has order n. If G has more than one component or block, then some component or end-block satisfies the hypotheses, and by induction it contains a subdivision of K_4. Hence we may assume that G is 2-connected. □



Proof.

If G has a cutset $S = \{u, v\}$, consider a lobe H that does not contain any vertex with degree less than 3. If this lobe contains another cutset of size 2, consider the new set and smaller lobe. In this way, we may assume H has no cutset of size 2. Then u and v both have at least two neighbors in H (else replacing one with its neighbor would yield another cutset in H). If $uv \in H$, then H satisfies the induction hypothesis. If not, then there is a $u - v$ path outside H, which can be treated as a subdivided edge.
If G is 3-connected, then for any vertex v, $G - v$ is 2-connected. Thus $G - v$ contains a cycle C. Then Lemma 29 says that there are three independent paths between v and C. These paths and C produce a subdivision of K_4. □

K_4 Subdivisions

Corollary

(Dirac [1964]) *If G has $m > 2n - 3$, then G contains a subdivision of K_4, and the graphs of size $2n - 3$ that don't contain a subdivision of K_4 are exactly the 2-trees.*

Proof.

Let G have $m > 2n - 3 = (3 - 1)\, n - \binom{3}{2}$. By Corollary 13, G contains a 3-core. By Theorem 30, it contains a subdivision of K_4. If a graph of size $2n - 3$ has no 3-core, it is maximal 2-degenerate. By Theorem 16, exactly the 2-trees do not contain a subdivision of K_4. □

- It is natural to ask what forces a graph to contain a subdivision of K_5. Minimum degree 4 does not suffice, as shown by $K_{2,2,2}$.
- However, $m > 3n - 6$ forces G to contain a subdivision of K_5. This was conjectured by Dirac and proved by Mader [1998].

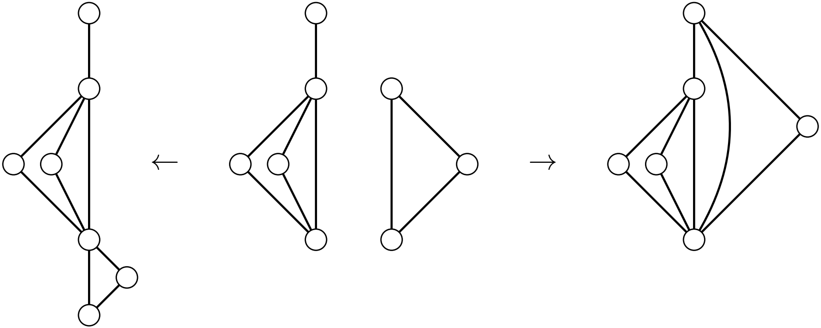
Series-Parallel Graphs

- Electrical circuits can be modeled using graphs, with edges representing wires, and vertices representing their intersections. Components of an electrical circuit can be combined in series (one after another) or in parallel (beside each other).

Definition

A **series-parallel graph** is a multigraph with two distinguished vertices (the **source** and **sink**) that can be constructed from K_2 using two operations:
parallel composition–identifying the sources and sinks of two series-parallel graphs
series composition–identifying the source of one series-parallel graph with the sink of another.

Example. Series composition (left) and parallel composition (right) of two series-parallel graphs.



Series-Parallel Graphs

- We could view the edges of a series-parallel graph as being directed from the source to the sink, but this is not required.

Theorem

(Duffin [1965]) *For a graph G, the following are equivalent:*
1. G contains no K_4-subdivision
2. G is contained in a 2-tree
3. every 2-connnected component of G is series-parallel.

- $3 \Rightarrow 2 \Rightarrow 1$ is not difficult, but the rest of the proof is harder.
- This implies that any series-parallel graph (not multigraph) has size at most $2n-3$, and 2-trees are the extremal graphs.

- There is a well-known characterization of planar graphs.

Theorem

(**Kuratowski's Theorem** | Kuratowski [1930]) *A graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.*

- This suggests results on K_5 subdivisions.

Theorem

(*Mader* [1998]) *If G has $m > 3n-6$, then G contains a subdivision of K_5.*
(Mader [2005]) *The graphs of size $3n-6$ that don't contain a subdivision of K_5 are exactly maximal planar graphs, and graphs formed by identifying triangles of maximal planar graphs.*

- Consider a graph in the plane with edges that are line segments with fixed length that are hinged at the vertices (the angles at the vertices may vary).

Definition

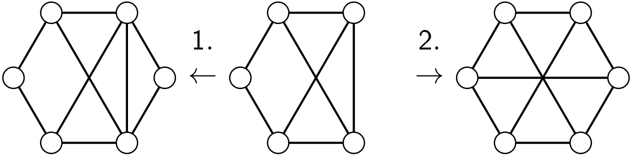
A graph is **rigid** if when its vertices are placed in general position in the plane (fixing the lengths of the edges), there is no movement of the graph in the plane preserving the edge lengths that does not also preserve all distances between vertices. A graph is **flexible** if it is not rigid. A graph is a **Laman graph** if and only if $m = 2n - 3$ and each nontrivial subgraph with order n' has size $m' \leq 2n' - 3$.

- Any rigid graph must be 2-connected, since multiple components could be moved independently, and multiple blocks could be rotated at a cut-vertex.
- Laman graphs include all maximal 2-degenerate graphs, and some with 3-cores.

- Henneberg found an operation characterization of rigid graphs.

Theorem

(Henneberg [1911]) *A graph is a minimal rigid graph if and only if it can be constructed by starting with K_2 and iterating the following two operations (**Henneberg operations**).*
1. Add a vertex of degree two.
2. Add a vertex of degree three adjacent to two vertices that are neighbors and delete the edge between them.



- Laman proved a characterization of rigid graphs involving their sizes.

Rigid Graphs

Theorem

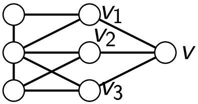
(Laman [1970]) *A graph has a Henneberg construction if and only if it is a Laman graph.*

Proof.

($\Rightarrow$) Assume G has a Henneberg construction. Certainly K_2 is a Laman graph and both operations increase n by 1 and m by 2 in G. If a vertex is added to a subgraph of G, its size is increased by at most 2. Thus the operations preserve Laman graphs.
($\Leftarrow$) (HORSSSSSW [2005]) If $n = 2$, then K_2 is the only Laman graph. Assume that any Laman graph with order less than $n > 2$ has a Henneberg construction. If G is a Laman graph with order n, then $m = 2n - 3$, and it has a vertex of degree at most 3. If any vertex v has degree 0 or 1, then $G - v$ is not a Laman graph. If v has degree 2, then $G - v$ is a Laman graph since its size is $2(n-1) - 3$ and all its subgraphs are subgraphs of G. □

Proof.

If v has degree 3, let $N(v) = \{v_1, v_2, v_3\}$. Let $H = G - v$, which has order $n-1$, but only $2(n-1)-4$ edges. We must add one edge joining one of the three pairs of vertices in $N(v)$. Consider the rigid components of H: maximal subsets of some k vertices spanning $2k-3$ edges. Now v_1, v_2 and v_3 cannot belong to the same rigid component (otherwise the size restriction would be violated in G on the subset consisting of this component and v). Two rigid components share at most one vertex, otherwise their union would be a larger Laman subgraph. Say v_1 and v_2 are not in a common rigid component. Then adding $e = v_1 v_2$ doesn't violate the size restriction on any subset and converts H to a Laman graph H'. Then H' and hence G has a Henneberg construction. □



Definition

The **extremal graphs** for the bound $f(G) \leq g(G)$ are the graphs that make it an equality.

- **Extremal graph theory** is concerned with the question of finding the extreme value (maximum or minimum) of a parameter over some class of graphs. Perhaps the most common is example is finding the maximum size over a given class.

Definition

The **extremal number** $ex(n, \mathbb{G})$ is the maximum size among all graphs of order n that do not contain any graph $G \in \mathbb{G}$ as a subgraph. We write $ex(n, G)$ when $\mathbb{G} = \{G\}$.

Extremal Graph Theory

- Thus a graph with size $ex(n, \mathbb{G}) + 1$ must contain some $G \in \mathbb{G}$ as a subgraph. The following table lists some extremal numbers, along with the extremal graphs when known.

Class/Property	$ex(n, \mathbb{G})$ (large n)	Extremal Graphs
contains cycle	$n-1$	trees
not outerplanar	$2n-3$	maximal outerplanar
contains K_4 subdivision	$2n-3$	2-trees
nonplanar	$3n-6$	maximal planar
contains K_5 subdivision	$3n-6$	identify K_3s of max. plana
two disjoint cycles	$3n-6$	$K_3 + \overline{K}_{n-3}$
contains $k+1$-core	$kn - \binom{k+1}{2}$	maximal k-degenerate
has $\alpha(G) < n-k+1$	$kn - \binom{k+1}{2}$	$K_k + \overline{K}_{n-k}$
contains $(k+1)K_2$	$kn - \binom{k+1}{2}$	$K_k + \overline{K}_{n-k}$

Definition

The **independence number** $\alpha(G)$ of a graph G is the size of the largest independent set of G.

Proposition

If G has $\alpha(G) \leq n-k$, then $m \leq kn - \frac{k(k+1)}{2}$. The extremal graphs are $K_k + \overline{K}_{n-k}$.

Proof.

Let G have an independent set of $n-k$ vertices. Add all other possible edges. The other k vertices induce K_k, and the $n-k$ vertices are joined to all of them. This produces the graph $K_k + \overline{K}_{n-k}$, which is a k-tree. □

- The results for two disjoint cycles and matchings are more difficult to prove.

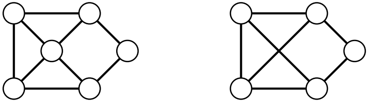
Uniquely Colorable Graphs

Definition

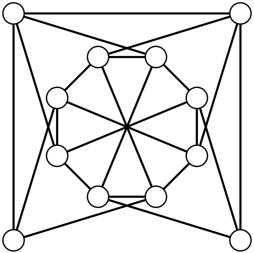
A graph is **uniquely** k-**colorable** if any k-coloring produces the same vertex partition. A graph is **uniquely colorable** if any minimum coloring produces the same vertex partition.

- Complete graphs are uniquely colorable.
- So are trees.
- If G is uniquely k-colorable, and a vertex v of degree $k-1$ is added so that it is adjacent vertices in all but one color class, the new graph is uniquely k-colorable.
- Uniquely colorable graphs include all k-trees, but not all maximal k-degenerate graphs.
- $k - trees \subset \begin{array}{c} uniquely\ colorable \\ maximal\ k - degenerate \\ graphs \end{array} \subset \begin{array}{c} maximal \\ k - degenerate \\ graphs \end{array}$

Uniquely Colorable Graphs



- A uniquely k-colorable graph G has $\delta(G) \geq k-1$.
- Let G be a uniquely k-colorable graph with $d(v) = k-1$ for some vertex v. Then $G - v$ is uniquely colorable.
- Thus uniquely $k+1$-colorable graphs of larger order can always be constructed by adding (uniquely colorable) maximal k-degenerate appendages.
- Do all uniquely $k+1$-colorable graphs contain a maximal k-degenerate graph? No!
- Surprisingly, there are also triangle-free uniquely 3-colorable graphs.



Lemma

In a uniquely colorable graph, any two color classes induce a connected graph.

Proof.

If not, the colors could be exchanged on one component of the graph they induce. □

Theorem

(Xu [1990]) *A uniquely $k+1$-colorable graph has $m \geq kn - \frac{k(k+1)}{2}$.*

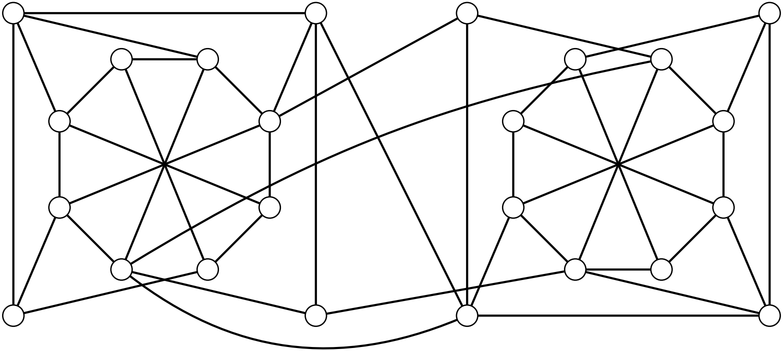
Proof.

Let G be uniquely $k+1$-colorable with color classes V_i. Each edge of G is in exactly one subgraph induced by two color classes. Thus

$$\begin{aligned} m(G) &= \sum_{i \neq j} m(G[V_i \cup V_j]) \\ &\geq \sum_{i \neq j} (|V_i \cup V_j| - 1) \\ &= \sum_{i \neq j} |V_i \cup V_j| - \binom{k+1}{2} \\ &= kn - \frac{k(k+1)}{2}. \end{aligned}$$

□

- Uniquely colorable maximal k-degenerate graphs are extremal graphs for this bound. The triangle-free graph above is not. Xu conjectured that any extremal graph must contain a triangle.
- (Akbari/Mirrokni/Sadjad [2001]) The following graph is uniquely 3-colorable, triangle-free, and has $m = 2n - 3$.



Uniquely Colorable Graphs

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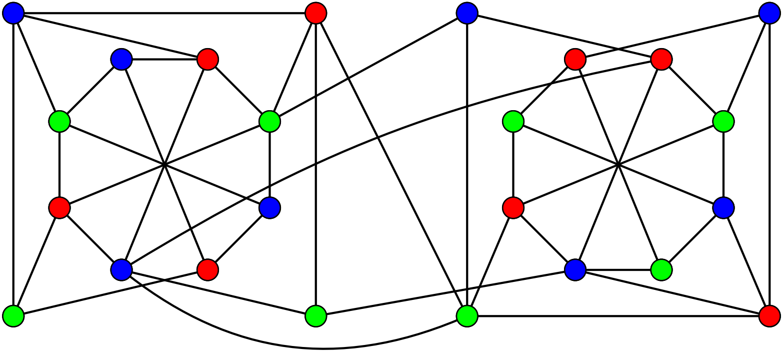

Apex Graphs

- What about graphs with $m = 4n - 10$?
- An **apex graph** G has $G - v$ planar for some vertex v.
- A vertex v has degree at most $n - 1$.
- The planar graph $G - v$ has size at most $3(n-1) - 6$.
- Thus an apex graph has size at most $3(n-1) - 6 + (n-1) = 4n - 10$.
- They have application in the proof of Hadwiger's Conjecture ($k = 6$), and other problems in graph theory.

Thank you!