Spanning 2-Trees of Maximal Planar Graphs

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What I Did in Summer 2021

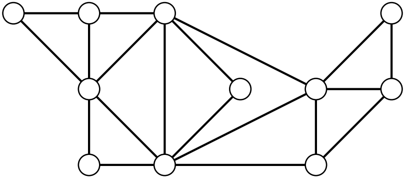
- I wrote a survey article on k-trees, maximal k-degenerate graphs, and related topics.
- 40 pages
- 268 references
- It was quite interesting to "read" everything written on a particular topic, and see how it developed over time.
- Some papers had a big impact, and have been cited repeatedly, while others seemed to attract no attention.
- Various conjectures and open problems are presented in the papers I read.
- Some were resolved in later papers, while others are still open.
- One particular open problem struck me, and I began working on it.
- First, some definitions.

Definitions

Definition

A k-**tree** is a graph that can be formed by starting with K_{k+1} and iterating the operation of making a new vertex adjacent to all the vertices of a k-clique of the existing graph. The clique used to start the construction is called the **root** of the k-tree.

- Note that a k-tree is a chordal graph. A more general recursive construction of k-trees is that K_{k+1} is a k-tree, and any larger k-tree can be formed by identifying two k-trees on K_k or K_{k+1}.



Definitions

Definition

A **spanning subgraph** of a graph has the same vertex set.

- Every connected graph has a spanning tree.

Definition

A **Hamiltonian cycle** of a graph G is a spanning cycle of G. A graph with a Hamiltonian cycle is called a **Hamiltonian graph**.
A graph is **planar** if it has a drawing in the plane that has no crossings. The **regions** of a plane drawing are the maximal pieces of the plane surrounded by edges and vertices. The infinite region is the **exterior region**. The **length** of a region is the length of a walk around it. A graph is **maximal planar** if no edge can be added without making it not planar.
A graph is **outerplanar** if it has a plane drawing with all vertices on the exterior region.

Related Results

- Bern [1] showed that determining whether a graph has a spanning k-tree is NP-complete when $k \geq 2$.
- Cai and Maffay [7] show this is true even for planar graphs with $\Delta(G) \leq 6$ when $k = 2$.
- Cai found several sufficient conditions for a spanning 2-tree and showed that it is NP-complete to determine if G has a spanning k-tree even given a spanning l-tree of G, $l < k$ [5, 6].
- Any complete graph has a spanning k-tree.
- Bern [1] showed that it is NP-complete to find a minimum spanning 2-tree for weighted complete graphs, and found an exponential algorithm for this problem.

Related Results

- Cai [5] shows that there is no good approximation algorithm for weighted complete graphs in general, but there is such an algorithm when they satisfy the triangle inequality.
- Shangin and Pardalos [7] consider various heuristics for the spanning k-tree problem.
- Ding [9] found applications of spanning k-trees to linguistic grammars, the RNA 3D structure prediction problem, and learning Markov or Bayesian networks.
- Spanning 2-trees have applications in geodesy (geodetic surveying) [3] and logic and probability [1, 4].
-
- Leizhen Cai [5, 6] asked in 1995 whether every maximal planar graph contains a spanning 2-tree.

Hamiltonian Cycles and 2-Trees

- Cai did not conjecture an answer, but I will reframe the problem as a conjecture to simplify discussion of it.

Conjecture

Every maximal planar graph contains a spanning 2-tree.

- It is easy to show that some special classes of maximal planar graphs have spanning 2-trees.

Lemma

[5] *Every Hamiltonian maximal planar graph contains a spanning 2-tree.*

Proof.

Adding the edges inside (or outside) a Hamiltonian cycle produces a spanning 2-tree. □

Hamiltonian Cycles and 2-Trees

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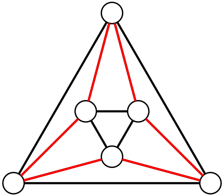
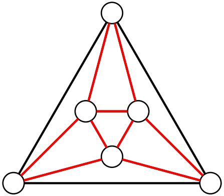




- Denote a 4-connected maximal planar graph as a **4MP**.
- For these graphs, the converse is true.

Proposition

Every spanning 2-tree of a 4MP has a unique Hamiltonian cycle.

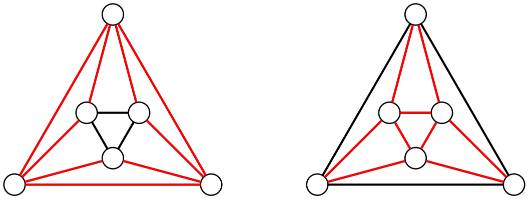
Proof.

A 2-tree has a Hamiltonian cycle if and only if it contains no $K_2 + \overline{K}_3$ [6]. Let G be a 4MP with a spanning 2-tree T. If T is not Hamiltonian, it contains $K_2 + \overline{K}_3$, so G has a separating triangle and is not 4-connected. Thus T has a Hamiltonian cycle C.
A 2-tree is Hamiltonian if and only if it is outerplanar [5]. It is easily shown by induction that the cycle is unique, and T can be drawn so that C is the exterior region. □

- This shows a correspondence between Hamiltonian cycles and pairs of spanning 2-trees of 4MPs.

Corollary

Every 4MP has twice as many spanning 2-trees as Hamiltonian cycles.



Hamiltonian Cycles and 2-Trees

- Whitney [9] showed that every 4MP is Hamiltonian.
- Tutte proved a stronger statement.

Theorem

[8] Every *planar 4-connected graph has a Hamiltonian cycle through any two edges of a region.*

- Cai observed the following corollary to Theorem 9.

Corollary

[5] *Every 4MP contains a spanning 2-tree.*

- There is another easy special case. Cai and Maffray [7] showed that every l-tree contains a spanning k-tree when $l \geq k \geq 1$.

Corollary

[7] *Every 3-tree contains a spanning 2-tree.*

4-blocks

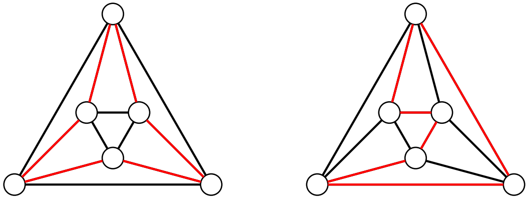
- Denote a 4MP or K_4 as a **4-block**.
- Every maximal planar graph can be formed by identifying triangles of 4-blocks.
- Any 4-block has a spanning 2-tree. The question is whether a spanning 2-tree for the whole graph can be pieced together from those of the 4-blocks.
- A spanning 2-tree contains 0, 1, 2, or 3 edges of any given triangle.
- If there were some 4MP such that for every spanning 2-tree T, there is some triangle with no edge of T, that would be sufficient to disprove Conjecture 4. (We could simply attach another 4-block at every triangle, and a spanning 2-tree could not extend into all of them.) We will show that this is the case.

4-blocks

Definition

A maximal planar graph has a **linear Hamiltonian cycle** if the regions inside (or outside) the cycle share edges with at most two other such regions (that is, the dual of these regions is a path).

- The Hamiltonian cycle on the right is linear; the one on the left is not.



Linear Hamiltonian Cycles

Conjecture

Every 4MP has a linear Hamiltonian cycle.

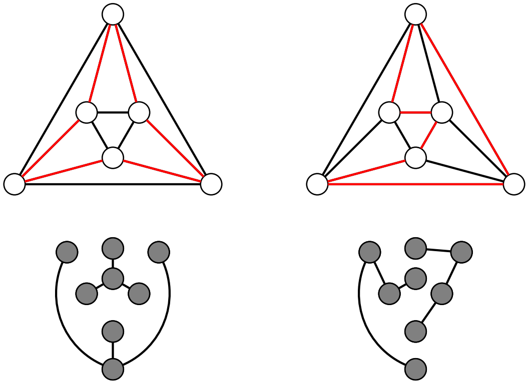
- Conjecture 13 is weaker than Conjecture 4. We will show that Conjecture 13 is false, so Conjecture 4 is false.
- To study Conjecture 13, it is convenient to look at the dual graph.

Definition

The **Hamiltonian dual** of a planar graph with a given Hamiltonian cycle is formed by deleting any edges that cross the Hamiltonian cycle from the dual graph.

Path-Tree Partitions

- Denote the dual of a 4MP as a **4MP dual**.
- A 4MP dual is a 3-connected cubic planar graph with no nontrivial 3-edge cut. (A trivial edge cut has all edges incident with a common vertex.)



Definition

A cubic graph has a **path-tree partition** if its vertices can be partitioned into two sets so that one induces a path and the other induces a tree. A **path-path partition** and **tree-tree partition** are defined similarly. A **Yutsis graph** is a multigraph in which the vertex set can be partitioned in two parts such that each part induces a tree.

- A tree-tree partition is also known as a **Yutsis decomposition**. Yutsis graphs have applications in physics, particularly the quantum theory of angular momenta [10].
- A simple degree sum argument shows that the two vertex sets in such a partition must have equal size.

- Theorem 9 implies that the every 4MP dual has a tree-tree partition.

Proposition

A 4MP has a linear Hamiltonian cycle if and only if its dual has a path-tree partition.

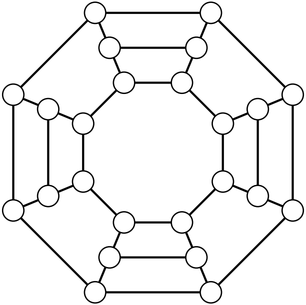
Proof.

($\Rightarrow$) Since all vertices are on a Hamiltonian cycle, no vertex is inside it. Thus each component of the Hamiltonian dual is acyclic. Each is clearly connected, so each is a tree. They have the same order since there are the same number of regions inside and outside the Hamiltonian cycle. To have a linear Hamiltonian cycle, one of the trees must be a path.
($\Leftarrow$) If a 4MP dual has a path-tree partition, the 4MP clearly has a linear Hamiltonian cycle. $\square$

Path-Tree Partitions

- To show that Conjecture 13 is false, we produce a 4MP dual that has no path-tree partition.

Let G_k have vertices a_i, b_i, c_i, $1 \leq i \leq 2k$. The a_i's and b_i's induce $2k$-cycles, $a_i \leftrightarrow c_i$, $b_i \leftrightarrow c_i$, and $c_{2i-1} \leftrightarrow c_{2i}$ for all i, (all mod $2k$). We show G_4 below.



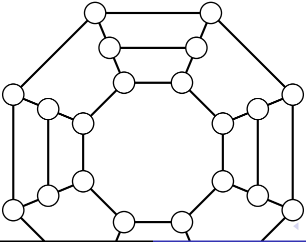
Path-Tree Partitions

Theorem

For $k \geq 4$, G_k has no path-tree partition.

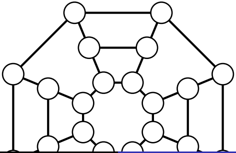
Proof.

Denote the graph formed from C_6 by adding a single chord joining opposite vertices as a **brick**. Clearly G_k contains k bricks. Assume to the contrary that G_k has a path-tree partition with path P and tree T. Both P and T must contain at least one vertex from each cycle. □



Proof.

The ends of P are in one or two bricks, so P must pass through at least two bricks without ending. It is not possible for P and T to both pass through the same brick since the two c-vertices would be part of an induced 4-cycle of one of them. If P enters and exits a brick using two a's, it must end at a b in the same brick (or vice versa).
To pass through a brick without ending there, P must enter at an a-vertex and exit at a b-vertex (or vice versa). Now P must contain (nonadjacent) a-vertices in distinct bricks with a b-vertex between them. But then the graph induced by the vertices not in P is disconnected and hence not a tree, a contradiction. □



Path-Tree Partitions

- Essentially the same construction appeared in [4], where it is used to analyze the number of triangles of certain types in Hamiltonian maximal planar graphs.

Corollary

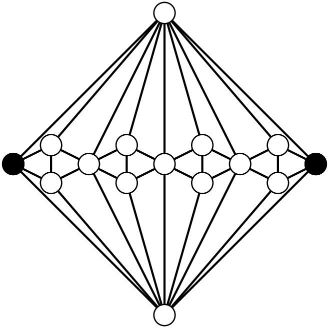
For each orientable or non-orientable surface there are triangulations without spanning 2-trees.

Proof.

Take an arbitrary triangulation of the surface and glue a copy of the plane triangulation without spanning 2-tree into a facial triangle. □

- Note that G_4 has order 24. In fact, a computer search conducted for [4] has shown that the smallest order of a 4MP dual with no path-tree partition is 24 (Gunnar Brinkmann).

- The dual of G_4 is a maximal planar graph with order 14. (See below, where the two black vertices must be identified.)
- Adding a degree 3 vertex inside each of its 24 regions produces a maximal planar graph of order 38 with no spanning 2-tree.

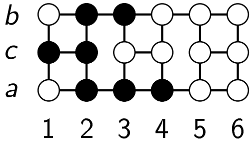


Theorem

The graph G_4 has no tree-tree partition so that one of the trees has branches only at c-vertices.

Proof.

Call the trees T_1 and T_2, where T_2 has branches only at c-vertices. If T_2 has no branches, we would have a path-tree partition, which is impossible by Theorem 17. Denote the vertices using the grid in the figure below, and let the vertices in T_2 be black. □



Path-Tree Partitions

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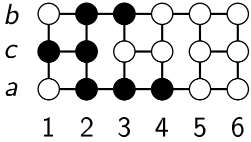
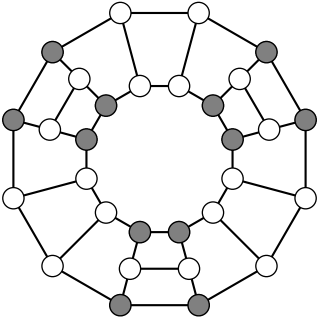


Proof.

Assume such a partition is possible, and WLOG, assume T_2 has a branch at c_2. Then $\{c_1, a_2, b_2\} \subseteq V(T_2)$, so $\{a_1, b_1\} \subseteq V(T_1)$. The only way for T_2 to pass through the brick to the right without disconnecting T_1 is for it to contain a_3, b_3 and one of a_4 or b_4 (say a_4). However, now T_2 cannot pass through the following brick without disconnecting T_1, which is impossible. Thus no such partition exists.

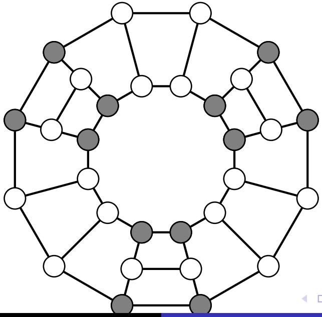
- Thus in the dual of G_4, adding vertices inside the triangles corresponding to a- and b-vertices is sufficient to produce a maximal planar graph with no spanning 2-tree. This graph has order 30. This is not the smallest order of such a graph.
- We can find a smaller maximal planar graph with no spanning 2-tree by reducing the number of vertices added to the large 4-block. Consider the graph H_{30} below.



Theorem

The graph H_{30} has no tree-tree partition so that one of the trees has no branches at the gray vertices.

- The dual of H_{30} has order 17. Adding degree 3 vertices in the 12 regions that correspond to the gray vertices, we find a maximal planar graph of order 29 with no spanning 2-tree.
- It is unclear whether this is the smallest order of such a graph.



- To produce infinite classes of graphs that do and don't have path-tree partitions, we need an operation to generate cubic graphs.

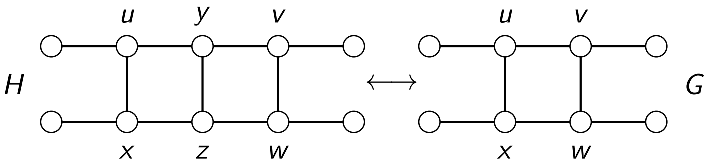
Definition

Let uv and wx be edges of a cubic graph. Let **adding a handle** be the operation of subdividing uv and wx and adding a new edge yz between the new vertices. Let **4-handling** be the operation of adding a handle between two nonadjacent edges of a region of length 4 of a 4MP dual.

- Every 4MP dual can be constructed from the cube by adding handles [10, 2].

Proposition

Let H be formed by 4-handling a 4MP dual G. If G has no path-tree partition then so does H.



Proof.

(contrapositive) Let $uyvwzx$ be a 6-cycle of H, and yz be a handle. Let $uvwx$ be a region of length 4 of G which is 4-handled to produce the 6-cycle. Assume H has a path-tree partition with path P and tree T. We want to show that G has a path-tree partition with path P' and tree T'.

□

Proof.

First suppose that yz is not in P or T. Then y and z are not both in P or both in T. If uy and yv are both in P or T, let uv be in P' or T', respectively. Similarly, if wz and zx are both in P or T, let wx be in P' or T', respectively. Else don't put uv (or wx) in P' or T'. Thus P' and T' are both connected, acyclic, and have the same order in G, so we have a path-tree partition of G.
Now suppose that yz is in T. At least one of the other vertices, say x, is in T. Then u is a leaf of P. Then put ux in T' and leave v and w in the same corresponding sets. Thus P' and T' are both connected, acyclic and have the same order in G, so we have a path-tree partition of G. If we exchange the roles of P and T, the argument is similar. □

Path-Path Partitions

- The statement of this proposition does not hold in general for regions of length more than 4.
- Since all graphs formed by 4-handling G_4 have no path-tree partition, we have an infinite class of counterexamples to Conjecture 4.

Theorem

There are maximal planar graphs so that any 2-tree that is a subgraph of a graph G with order n contains at most $\frac{6n+23}{7}$ vertices.

- The proof uses the graphs G_k.

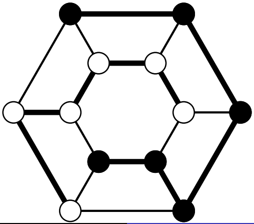
Path-Path Partitions

- The smallest 4MPs are the double wheels $C_{n-2} + \overline{K}_2$, whose duals are the prisms $C_r \Box K_2$, $r \geq 4$.
- Note that any prism can be generated from the cube by adding handles.

Theorem

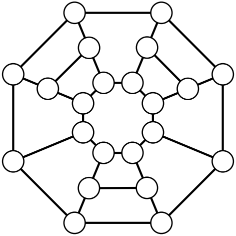
Any 4MP dual constructed from a prism by adding at most two handles has a path-path partition.

- The proof (which requires many tedious cases) is omitted.
- A prism has a path-path partition using any two spokes.



Path-Path Partitions

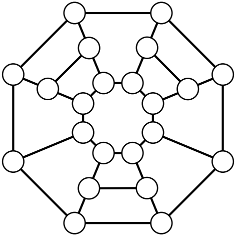
- Note that G_4 can be generated from a prism by adding four handles.
- There is a 4MP dual of order 22 with no path-path partition. It can be formed by adding three handles to a prism.
- Call the graph below H_{22}.



Theorem

The graph H_{22} has no path-path partition.

Path-Path Partitions



Theorem

The graph H_{22} has no path-path partition.

- A computer search by Gunnar Brinkmann shows that 22 is the smallest order of a 4MP dual with no path-path partition (personal communication).
- I had hand-checked that all 4MP duals of order at most 16 have a path-path partition.

The Structure of Spanning Subgraphs

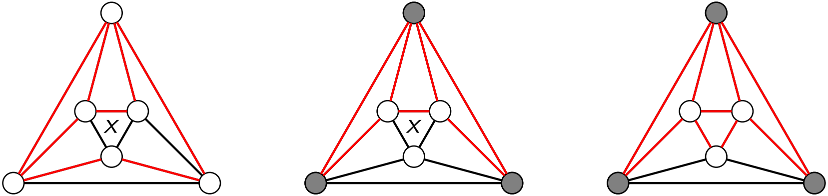
- Next we consider how a spanning 2-tree of a maximal planar graph can constructed from spanning 2-trees of 4-blocks.

Lemma

Let G be a graph with a (not necessarily spanning) 2-tree T and t a separating triangle in G. Let C_0 and C_1 be two components of $G - t$, so that t contains vertices of C_1. Let G_0 be the graph induced by $C_0 \cup t$. Then there is a 2-tree T_0 of G_0 with vertex set the vertices of the restriction of T to G_0 together with the vertices of t and edge set the edges of the restriction of T to G_0 - among them at least one edge of t - and possibly some edges of t not contained in T.
This 2-tree T_0 is unique and we will refer to it as the ***induced 2-tree****.*

The Structure of Spanning Subgraphs

- Consider the triangulation that identifies the two triangles marked x. A spanning 2-tree is in red. The induced 2-tree for the central 4-block is shown at right.



- Using the previous lemma, a large class of triangulations can be shown to have spanning 2-trees. Their 4-blocks are “good 4-blocks”, which allow the spanning 2-tree to be constructed by adding one 4-block at a time.
- An exhaustive search shows that all 4-blocks of order 9 are good. There is a non-good 4-block with order 10.

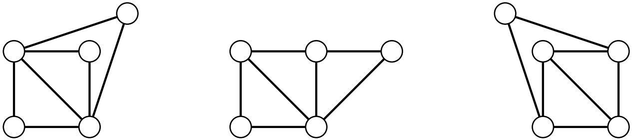
The Structure of Spanning Subgraphs

- While Conjecture 4 is false, a weaker statement is true.

Definition

A graph is k-**degenerate** if its vertices can be successively deleted so that when deleted, each has degree at most k. A graph is **maximal** k-**degenerate** if no edges can be added without violating this condition.

Example. The three maximal 2-degenerate graphs of order 5 are shown below.



- Every k-tree is maximal k-degenerate, but the converse is false when $k > 1$.
- We construct a maximal k-degenerate graph (or k-tree) by starting with K_k and successively adding vertices of degree k.

Theorem

Every maximal planar graph contains a spanning maximal 2-degenerate graph.

The Structure of Spanning Subgraphs

Theorem

Every maximal planar graph contains a spanning maximal 2-degenerate graph.

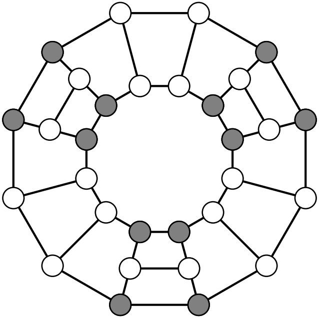
Proof.

This is obvious for order $n \leq 3$. Let G be maximal planar, and construct it by starting with some 4-block B_1 and iteratively adding each new 4-block B_r by identifying a triangle T_r of B_r with a triangle T_r^* of the existing graph. Let $G_1 = B_1$ and G_r be the graph after r 4-blocks have been added. We will show that for each r, G_r has a spanning maximal 2-degenerate subgraph M_r.

If B_r is a 4-block containing triangle T_r, then by Theorem 9 it has a spanning 2-tree that contains T_r. Any 2-tree can be constructed starting with any of its triangles. When identifying T_r and T_r^*, delete any edges of T_r from the spanning 2-tree of G_r that are not in M_{r-1}. Thus the construction of M_{r-1} can continue from T_r into G_r, producing M_r. Iterating this process proves the theorem. □

Conclusion

- There is more work to do on this problem.
- The big question is to characterize which maximal planar graphs have a spanning 2-tree. That problem seems difficult.
- Another question is whether 29 is the smallest order of a maximal planar graph with no spanning 2-tree.



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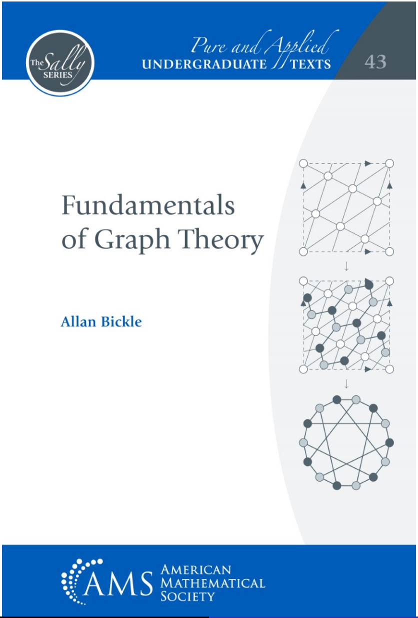
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