Properties of Sierpinski Triangle Graphs

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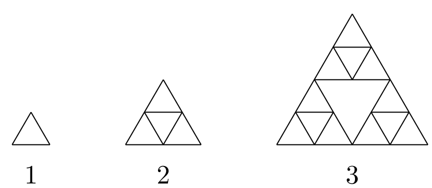
Abstract

The Sierpinski triangle can be modeled using graphs in two different ways, resulting in classes of graphs called Sierpinski triangle graphs and Hanoi graphs. The latter are closely related to the Towers of Hanoi problem, Pascal's triangle, and Apollonian networks. Parameters of these graphs have been studied by several researchers. We determine the number of Eulerian circuits of Sierpinski triangle graphs and present a significantly shorter proof of their domination number. We also find the 2-tone chromatic number and the number of diameter paths for both classes, generalizing the classic Towers of Hanoi problem.

The Sierpinski triangle is a familiar fractal. One way to iteratively construct it is to start with a triangle (level 1). In each step, combine three copies of level k together to produce level $k+1$.

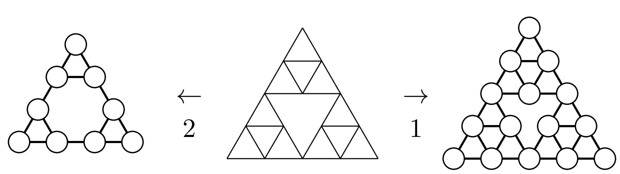


There are two ways to model the Sierpinski triangle as a graph.

In Model 1, each intersection of lines is represented by a vertex, and each line segment between vertices is represented by an edge.

In Model 2, each copy of level 1 of the fractal is represented by a vertex, and there are edges between vertices that have a point in common.

The graphs in Model 2 can be considered duals of sorts for the graphs in Model 1.



Undefined notation and terminology will generally follow [1].

1 Sierpinski Triangle Graphs

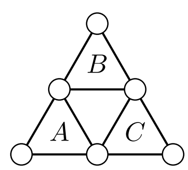
The graphs in Model 1 are known as **Sierpinski triangle graphs**. Denote the level k Sierpinski triangle graph ST_k. Thus $ST_1 = K_3$.

The recursive construction naturally leads to recurrence relations for many parameters. A recurrence for order is $n_1 = 3$, $n_{k+1} = 3n_k - 3$, with solution $n_k = \frac{3}{2}3^{k-1} + \frac{3}{2}$. The size of ST_k is clearly $m_k = 3^k$. Note that ST_k has 3 degree 2 vertices and $\frac{3}{2}3^{k-1} - \frac{3}{2}$ degree 4 vertices. Denote the degree 2 vertices of ST_k as **corners**, and the three vertices contained in two copies of ST_{k-1} as **middle vertices**.

Many researchers have determined properties of these graphs. Hinz, Klavzar, and Zemljic [5] have a survey of Sierpinski triangle graphs and related concepts. Bradley [3] found that for $k \geq 3$, There are $3^{\frac{3^{k-2}-3}{2}} 2^{3^{k-2}}$ Hamiltonian cycles in ST_k.

Since ST_k is connected with all even degrees, it is Eulerian. We can count the number of Eulerian circuits, considered as ordered lists of edges without regard to starting vertex or direction. This sequence begins 1, 16, 65536, 4503599627370496 ...

Theorem 1.1. *There are $4^{3^{k-1}-1}$ Eulerian circuits in ST_k.*



Proof. Let E_k be the number of Eulerian circuits of ST_k. Clearly $E_1 = 1$. Consider ST_{k+1}, which is formed from three copies of ST_k, which we denote A, B, and C. The circuit can be split into segments based on which of A, B, or C contain each edge. We split it each time the circuit reaches a middle vertex. Then there are exactly two segments for each of A, B, and C. The segments may occur in a consistent order (e.g. $ABCABC$) or there may be a reversal (e.g. $ABCCBA$) where the circuit touches a middle vertex but continues in the same copy of ST_k. There cannot be more than one reversal, since then the circuit would not be Eulerian.

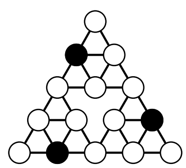
If there is a reversal, there are three middle vertices where it may occur. Without loss of generality, assume the circuit has the form $ABCCBA$. The segments of the circuit in A correspond to an Eulerian circuit of A. To construct an Eulerian circuit of G, we start with A and choose a trail to B in $2E_k$ ways. Similarly, there are $2E_k$ trails through B and C, after which the rest of the circuit is forced. There are two directions that could be taken, so divide by 2 to find $4\left(E_k\right)^3$ choices for each of the three reversals.

If there is no reversal, we similarly have $2E_k$ choices for each subgraph, and two directions, so there are $4\left(E_k\right)^3$ choices. Thus $E_{k+1} = 3 \cdot 4\left(E_k\right)^3 + 4\left(E_k\right)^3 = 16\left(E_k\right)^3$.

It is easily checked that $E_k = 4^{3^{k-1}-1}$ is the solution to this recurrence relation. □

A **dominating set** of a graph G is a set S of vertices so that every vertex not in S is adjacent to a vertex in S. The **domination number** $\gamma(G)$ is the minimum size of a dominating set of G. Teguia and Godbole [10] showed that the domination number $\gamma(ST_k) = 3^{k-2}$ for $k \geq 3$. Their proof is about 1.5 pages. A much shorter proof of this uses a discharging-type argument.

Proposition 1.1. *[10] We have $\gamma(ST_k) = 3^{k-2}$ for $k \geq 3$.*



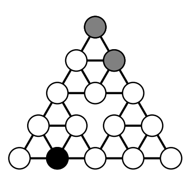
Proof. Clearly $\gamma(ST_3) \leq 3$. Also, $\gamma(ST_{k+1}) \leq 3\gamma(ST_k)$, so $\gamma(ST_k) \leq 3^{k-2}$ for $k \geq 3$.

For a lower bound, consider the 3^{k-2} copies of ST_2 in ST_k. Any minimum dominating set S of ST_k contains at least one degree 4 vertex or at least two corners of every ST_2. Assign each copy of ST_2 a score of 1 for each degree 4 vertex in S and $\frac{1}{2}$ for each corner in S. Let t be the total score. Then each ST_2 gets a score of at least 1, so $\gamma(ST_k) = |S| \geq t \geq 3^{k-2}$. □

Teguia and Godbole [10] showed that $diam(ST_k) = 2^{k-1}$. We can determine which pairs of vertices achieve this maximum.

Proposition 1.2. *For $k \geq 1$, $diam(ST_k) = 2^{k-1}$, and for $k \geq 2$, the pairs of vertices at distance 2^{k-1} are those on different exterior sides of the graph such that their distances to the closest middle vertices sum to at least 2^{k-2}.*

For example, the vertices at maximum distance from the black vertex are colored gray.



Proof. This is obvious for ST_1 and ST_2. For ST_k, let u, v be vertices at maximum distance. A geodesic between them must go through a middle vertex w. Now

$$d(u,v) \le d(u,w) + d(w,v) \le 2^{k-2} + 2^{k-2} = 2^{k-1}.$$

For this to be an equality, u and v must be on the exterior sides of their copies of ST_{k-1}. Thus they are on opposite exterior sides of ST_k. If the sum of their distances to the closest middle vertices is less than 2^{k-2}, there is a shorter path through the third copy of ST_{k-1}. □

Corollary 1.1. *For $k \ge 2$, there are $3\left(2^{k-2}+1\right)2^{k-2}$ pairs of vertices at distance 2^{k-1} in ST_k.*

Proof. There are three pairs of corners and three pairs of corner and middle vertex. Consider a vertex u that is distance r from the nearest middle vertex. Then the other end v of a maximum distance $u-v$ path must be at least $2^{k-2} - r$ from its nearest middle vertex. Thus there are $r+1$ choices for v. By symmetry, there are

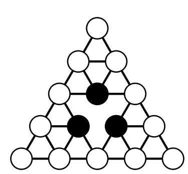
$$3+3+6\sum_{r=1}^{2^{k-2}-1}(r+1) = 6\sum_{r=1}^{2^{k-2}} r = 6\frac{1}{2}\left(2^{k-2}+1\right)2^{k-2} = 3\left(2^{k-2}+1\right)2^{k-2}$$

such pairs. □

We can also determine the radius of ST_k.

Proposition 1.3. *For $k \ge 3$, $rad\left(ST_k\right) = 3 \cdot 2^{k-3}$.*

Proof. Let v be a vertex of ST_k, and u be a vertex at maximum distance from v. Then u is in a different copy of ST_{k-1} from v. Let w be a middle vertex contained in the copies of ST_{k-1} containing u and v. Then $d(v,u) = d(v,w) + d(w,u) \ge \frac{1}{2}2^{k-2} + 2^{k-2} = 3 \cdot 2^{k-3}$. All three vertices that are midpoints of the geodesic paths between the middle vertices of ST_k achieve this minimum. □



A graph is **uniquely 3-colorable** if its vertex set has a unique partition into 3 independent sets. Klavzar [7] proved that ST_k is uniquely 3-colorable. We include the proof of this result since it is essential to the corollary that follows it.

Proposition 1.4. *[7] For $k \geq 1$, ST_k is uniquely 3-colorable.*

Proof. We use induction on the assumption that ST_k is uniquely 3-colorable and has three different colors on its corners. For ST_1, this is obvious; assume it holds for ST_{k-1}. We construct ST_k from three copies of ST_{k-1}. Thus the middle vertices of ST_k have three distinct colors. Thus the colorings of each copy of ST_{k-1} are forced, and there are three different colors on the corners of ST_k. □

In fact, ST_k is critical with respect to this property–deleting any edge results in a graph that is not uniquely 3-colorable.

Corollary 1.2. *For $k \geq 1$, ST_k is critical with respect to the property of being uniquely 3-colorable.*

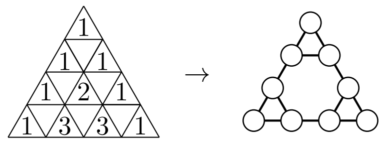
Proof. For ST_1, deleting any edge e results in a 2-colorable graph. When constructing $ST_2 - e$, this allows a coloring with a common color on two corner vertices. Similarly, there is a coloring of $ST_k - e$ with a common color on two corner vertices. □

2 Hanoi Graphs

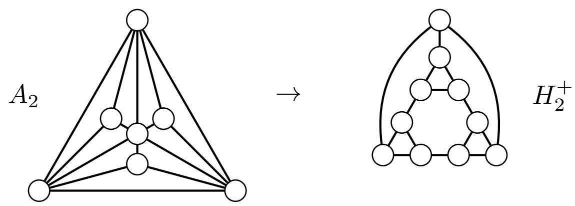
The graphs in Model 2 are known as **Hanoi graphs**. Denote the level k Hanoi graph H_k, so $H_1 = K_3$. The order of H_k is clearly $n_k = 3^k$. A recurrence for size is $m_1 = 3$, $m_{k+1} = 3m_k + 3$, with solution $m_k = \frac{3}{2}\left(3^k - 1\right)$. Note that H_k has 3 degree 2 vertices (**corners**) and $3^k - 3$ degree 3 vertices. Denote the three edges joining two copies of H_{k-1} as **middle edges**. Note that contracting every edge of H_k that is not on a triangle produces a Sierpinski triangle graph ST_{k-1}.

Hanoi graphs get their name from the Towers of Hanoi problem. In this problem, there are three pegs and k disks of different sizes, and a larger disk cannot be placed on a smaller disk. One disk at a time can be moved to another peg. This can be modeled with a graph. Label the pegs 0, 1, and 2 and assign each state a string indicating which peg contains the disks from smallest to largest. An edge joins two vertices when there is a valid move between the pegs. The result is a Hanoi graph.

Hanoi graphs also appear in Pascal's Triangle. Let vertices represent each binomial coefficient $\binom{r}{x}$ with $x < 2^k$ whose value is odd. Add edges between $\binom{r_1}{x_1}$ and $\binom{r_2}{x_2}$ if $r_1 = r_2$ or $r_1 = r_2 - 1$ and $|x_1 - x_2| \leq 1$. The result is the Hanoi graph H_k.



An **Apollonian network** is a planar 3-tree. Consider a particular class of Apollonian networks A_k, where $A_0 = K_3$, and A_{k+1} is formed by adding degree 3 vertices in all bounded triangular regions of A_k. Now the weak dual of A_k (excluding the outside region) is H_k. The dual graph is called the **extended Hanoi graph** H_k^+, which can be formed from H_k by adding a degree 3 vertex adjacent to the three corners. Thus H_k^+ is cubic.



There is a survey of Hanoi graphs and related concepts in [5]. A graph is **uniquely 3-edge-colorable** if its edge set has a unique partition into 3 independent sets. Klavzar [7] proved that H_k is uniquely 3-edge-colorable using a bijection with 3-colorings of ST_k. A direct (inductive) proof is also possible.

Proposition 2.1. *[7] For all $k \geq 1$, H_k is uniquely 3-edge-colorable.*

Proof. We use induction on the hypothesis that H_k is uniquely 3-edge-colorable, and the colors not used on the three corners are distinct. For H_1, this is obvious. Assume this is true for H_k, and use three copies of H_k to construct H_{k+1}. Each pair of edges added to join these three graphs have distinct colors, so all three do. Thus the coloring of one copy of H_k forces the colorings of the other two copies of H_k. Also, the colors not used on the three corners are distinct. □

The solution to the classic Towers of Hanoi problem is that it takes $2^k - 1$ moves to transfer all k disks from one peg to another peg. In graph theory terms, this means that the distance between two corners of H_k is $2^k - 1$. More generally, it is easy to prove that $diam\,(H_k) = 2^k - 1$. We can determine which pairs of vertices achieve this maximum.

Theorem 2.1. *For $k \geq 1$, $diam\,(H_k) = 2^k - 1$, and for $k \geq 2$, the pairs of vertices at distance $2^k - 1$ are those on different exterior sides of the graph such that their distances to the closest corner vertices sum to at most 2^{k-1}.*

Proof. This is obvious for H_1 and H_2. For H_k, let u, v be vertices at maximum distance. A geodesic between them must go through a middle edge $e = xy$. Now

$$d(u,v) \leq d(u,x) + 1 + d(y,v) \leq \left(2^{k-1}-1\right) + 1 + \left(2^{k-1}-1\right) = 2^k - 1.$$

For this to be an equality, u and v must be on the exterior sides of their copies of H_{k-1}. Thus they are on opposite exterior sides of H_k. If the sum of their distances to the closest corner vertices is more than 2^{k-1}, there is a path through the third copy of H_{k-1} with length at most

$$2\left(2^{k-1}-1\right) - \left(2^{k-1}+1\right) + 1 + \left(2^{k-1}-1\right) + 1 = 2^k - 2,$$

which is shorter. □

Corollary 2.1. *For $k \geq 1$, there are $3\left(2^{2k-2}+2^{k-1}-1\right)$ pairs of vertices at distance $2^k - 1$ in H_k.*

Proof. There are three pairs of corners. Consider a vertex u that is distance r from the nearest corner vertex. Then the other end v of a maximum distance $u-v$ path must be at most $2^{k-1}-r$ from its nearest corner vertex. Thus there are $2^{k-1}-r+1$ choices for v. By symmetry, the number P_k of such pairs is

$$\begin{aligned} P_k =& 3 + 6 \sum_{r=1}^{2^{k-1}-1} \left(2^{k-1}-r+1\right) \\ =& 3 + 6\left(\left(2^{k-1}-1\right)\left(2^{k-1}+1\right) - \frac{1}{2}\left(2^{k-1}-1\right)\left(2^{k-1}\right)\right) \\ =& 3 + 6\left(2^{k-1}-1\right)\left(2^{k-2}+1\right) \\ =& 3\left(2^{2k-2}+2^{k-1}-1\right) \end{aligned}$$

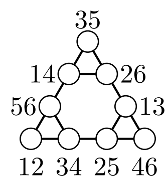
□

A **proper 2-tone coloring** of a graph G assigns two distinct colors to each vertex so that adjacent vertices have no common colors, and vertices at distance 2 have at most one common color. The **2-tone chromatic number** of G, $\tau_2(G)$, is the smallest k for which G has a proper t-tone k-coloring.

If H is a subgraph of G then $\tau_2(H) \leq \tau_2(G)$. Since $\tau_2(K_n) = 2n$, we have $\tau_2(G) \geq 2\omega(G)$. See [2] for basic information on 2-tone coloring.

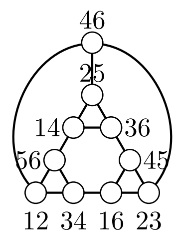
Proposition 2.2. *For $k \geq 1$, $\tau_2(H_k) = 6$.*

Proof. Since $K_3 \subseteq H_k$, we have $\tau_2(H_k) \geq \tau_2(K_3) = 6$. A 2-tone 6-coloring can be found by piecing together copies of H_2 with the coloring below. □

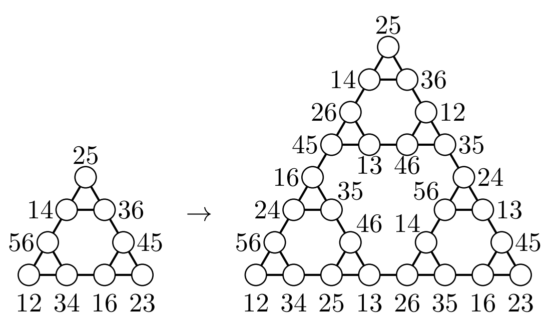


A somewhat more complicated argument is required to show that $\tau_2\left(H_k^+\right) = 6$ for all $k > 1$.

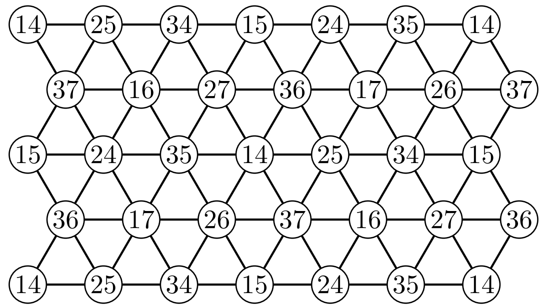
Theorem 2.2. *For $k \geq 2$, $\tau_2\left(H_k^+\right) = 6$.*



Proof. For $k = 2$, the coloring above works. For larger values of k, we use three copies of this coloring, but permute the colors to avoid conflicts. Clearly, permuting colors within a copy of H_k cannot create a conflict. In each copy of H_k, swap the pairs of colors in the left and top of the K_3 containing the corner. This maintains the same labels on the K_3s containing the corners and does not create any conflicts between the three H_ks. This is illustrated for H_3 below. The label 46 can be added to the extra vertex. □



The 2-tone chromatic number of Sierpinski triangle graphs is 7 for $k \geq 2$. The infinite triangular grid has 2-tone chromatic number 7, and Sierpinski triangle graphs are subgraphs of it. The coloring below can be extended infinitely since the boundaries use the same colors.



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