THE WALLIS PRODUCT FOR $\frac{\pi}{2}$

While π is an irrational number, there are many ways to express it using only rational numbers. There are several infinite series that converge to π. There is also an infinite product that converges to π. We first consider the integral $S_n = \int_0^\pi \sin^n x\, dx$. We find a reduction formula using integration by parts with

$$\begin{aligned} u &= \sin^{n-1} x & dv &= \sin x\, dx \\ du &= (n-1)\sin^{n-2} x\, \cos x\, dx & v &= -\cos x \end{aligned}$$

$$\begin{aligned} S_n &= \int_0^\pi \sin^n x\, dx \\ &= -\sin^{n-1} x\, \cos x\Big|_0^\pi - \int_0^\pi -\cos x\,(n-1)\sin^{n-2} x\, \cos x\, dx \\ &= 0 + (n-1)\int_0^\pi \left(1 - \sin^2 x\right)\sin^{n-2} x\, dx \\ &= (n-1)\int_0^\pi \sin^{n-2} x\, dx - (n-1)\int_0^\pi \sin^n x\, dx \\ n \cdot S_n &= (n-1)\, S_{n-2} \\ S_n &= \frac{n-1}{n} S_{n-2} \end{aligned}$$

Thus we have a recursive definition of the sequence S_n. The initial conditions are $S_0 = \int_0^\pi dx = \pi$ and $S_1 = \int_0^\pi \sin x\, dx = -\cos x|_0^\pi = 2$. The value of S_n depends whether n is even or odd.

$$S_{2n} = \frac{2n-1}{2n} S_{2n-2} = \frac{2n-1}{2n}\frac{2n-1}{2n}\frac{2n-1}{2n}\cdots\frac{5}{6}\frac{3}{4}\frac{1}{2}S_0 = \pi\prod_{k=1}^{n}\frac{2k-1}{2k}$$

$$S_{2n+1} = \frac{2n}{2n+1} S_{2n-1} = \frac{2n}{2n+1}\frac{2n-2}{2n-1}\frac{2n-4}{2n-3}\cdots\frac{6}{7}\frac{4}{5}\frac{2}{3}S_1 = 2\prod_{k=1}^{n}\frac{2k}{2k+1}$$

For $0 \le x \le \pi$, $\sin^{2n+1} x \le \sin^{2n} x \le \sin^{2n-1} x$, so

$$S_{2n+1} \le S_{2n} \le S_{2n-1}$$

$$1 \le \frac{S_{2n}}{S_{2n+1}} \le \frac{S_{2n-1}}{S_{2n+1}} = \frac{2n+1}{2n}$$

By the Squeeze Theorem,

$$1 = \lim_{n\to\infty}\frac{S_{2n}}{S_{2n+1}} = \frac{\pi}{2}\lim_{n\to\infty}\prod_{k=1}^{n}\frac{2k-1}{2k}\frac{2k+1}{2k} = \prod_{k=1}^{\infty}\frac{4k^2-1}{4k^2}$$

Thus the Wallis Product for $\frac{\pi}{2}$ is

$$\frac{\pi}{2} = \prod_{k=1}^{\infty}\frac{4k^2}{4k^2-1} = \frac{4}{3}\cdot\frac{16}{15}\cdot\frac{36}{35}\cdot\frac{64}{63}\cdots$$

The Wallis Product has a connection to Stirling's approximation for $n!$. Using calculus, we can show that $n! \approx C\sqrt{n}\left(\frac{n}{e}\right)^n$ for some constant C. First note that

$$\prod_{k=1}^{n}(2k-1) = (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1\frac{(2n)(2n-2)\cdots 4\cdot 2}{(2n)(2n-2)\cdots 4\cdot 2} = \frac{(2n)!}{2^n n!}$$

Let $n! = a_n\sqrt{n}\left(\frac{n}{e}\right)^n$, where $a_n \to C$. Then

$$\begin{aligned}
\frac{\pi}{2} &= \lim_{n\to\infty}\prod_{k=1}^{n}\frac{2k}{2k-1}\frac{2k}{2k+1} \\
&= \lim_{n\to\infty}\frac{2^n n!}{\frac{(2n)!}{2^n n!}}\frac{2^n n!}{(2n+1)\frac{(2n)!}{2^n n!}} \\
&= \lim_{n\to\infty}\frac{(2^n n!)^4}{(2n+1)\left((2n)!\right)^2} \\
&= \lim_{n\to\infty}\frac{\left(2^n a_n\sqrt{n}\left(\frac{n}{e}\right)^n\right)^4}{(2n+1)\left(a_{2n}\sqrt{2n}\left(\frac{2n}{e}\right)^{2n}\right)^2} \\
&= \lim_{n\to\infty}\frac{n^2 a_n^4}{(2n+1)\,2n\cdot a_{2n}^2} \\
&= \frac{C^4}{4C^2}
\end{aligned}$$

Thus $C = \sqrt{2\pi}$, so

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Specifically, this means

$$\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$