1 The Pigeonhole Principle

The Pigeonhole Principle is a simple, but surprisingly useful idea in combinatorics.

Theorem 1. (**The Pigeonhole Principle**) *If more than n objects that are distributed into n boxes, then some box has at least two objects.*

Strategy. The word 'some' indicates an existential quantifier. The contrapositive of the statement is "If every box has at most one object, then at most n objects that are distributed into n boxes." The proof of the statement is a simple consequence of a counting rule.

Proof. (contrapositive) If every box has at most one object, there are at most $n \cdot 1 = n$ objects in n boxes. □

Note that this is an existence proof. It does not tell us which box contains at least two objects. Nor does it give any idea how to find it, beyond checking them all.

To apply the Pigeonhole Principle, we must determine what are the objects and what are the boxes. Sometimes this is straightforward, but other times it can be tricky.

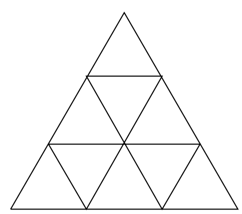
Example. Show that in a class of 32 people, at least two of them have a birthday on the same day of the month.

The boxes are the days of the month and the objects are the people. Since any month has at most 31 days, the Pigeonhole Principle implies that some day of the month is the birthday of at least two people.

There are many applications of the Pigeonhole Principle to geometry.

Example. Show that if ten points are inside an equilateral triangle with side length 1, some pair of them have distance at most $\frac{1}{3}$.

Solution. Divide the triangle into nine smaller equilateral triangles (see below). By the Pigeonhole Principle, at least two points are contained in one of the smaller triangles (including boundary). Now each smaller triangle has side length $\frac{1}{3}$. Thus some pair of points have distance at most $\frac{1}{3}$.



Example. [Putnam 2002] Show that for any five points on a sphere, there is a closed hemisphere that contains at least four of them.

Solution. A great circle of a sphere is a circle that divides it into two hemispheres. Any two points on a sphere determine a great circle of it. Use two of the five points to find a great circle. There are three other points. By the Pigeonhole Principle, at least two of them are contained in one of the two great circles. Thus at least four points are contained in a closed hemisphere.

The Pigeonhole Principle also has applications in number theory.

Example. Show that any set of $n+1$ integers from $[2n]$ contains a pair so that one divides another. Also show that this may not hold if only n integers are selected.

Strategy. The $n+1$ integers should be the objects, but it may not be immediately clear what the boxes are. When this happens, a good strategy may be to consider when the conclusion does not hold. Thus we look for a set of n integers from $[2n]$ so that none divide each other. Even numbers? Odd numbers? The first n? The last n? Yes, that's it. Further examination reveals that this is not the only possible answer. Replacing an even number with half of it also works. This suggests partitioning the set by the odd parts of the numbers. (The **odd part** of a number is its largest odd factor.)

Proposition 2. *Any set of $n+1$ integers from $[2n]$ contains a pair so that one divides another.*

Proof. Any integer a can be factored as $a = 2^p b$, b odd. Partition $[2n]$ into sets based on the odd part b. There are exactly n odd integers in $[2n]$, so there are n sets in the partition. Then given $n+1$ numbers from $[2n]$, two have the same odd part, so one is a multiple of the other. □

Another application of the Pigeonhole Principle is to sequences or lists of numbers. A **subsequence** of a sequence has some of the same numbers in the same order. A sequence is **monotone** if it is either increasing or decreasing, but not both.

Example. Consider the sequence 7, 3, 5, 1, 9, 4, 10, 2, 8, 6. Label each number with an ordered pair indicating the length of the longest increasing and decreasing subsequences ending with it.

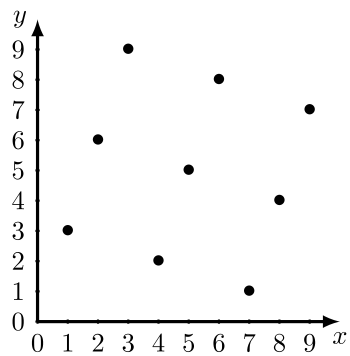
7	3	5	1	9	4	10	2	8	6
$(1,1)$	$(1,2)$	$(2,2)$	$(1,3)$	$(3,1)$	$(2,3)$	$(4,1)$	$(2,4)$	$(3,2)$	$(3,3)$

The longest increasing subsequence is 3, 5, 9, 10, and the longest decreasing subsequence is 7, 5, 4, 2. There is no monotone subsequence of length five.

Theorem 3. (**Erdos-Szekeres Theorem** | Erdos/Szekeres [1935]) *Any sequence of more than n^2 distinct numbers has a monotone subsequence of length more than n.*

Proof. (Seidenberg [1959]) Label each number with an ordered pair indicating the length of the longest increasing and decreasing subsequences ending with it. Assume to the contrary that there is no monotone subsequence of length more than n. Then there are at most n^2 possible pairs, so by the Pigeonhole Principle, some pair occurs twice. Let a_i and a_j, $i < j$, have the same pair. But if $a_i < a_j$, a_j would have a longer increasing subsequence, and if $a_i > a_j$, a_j would have a longer decreasing subsequence. This contradiction implies that there is a monotone subsequence of length more than n. □

The original proof of this theorem using induction was about two pages long. The theorem is sharp. For example, the sequence 3, 6, 9, 2, 5, 8, 1, 4, 7 of length nine has no monotone subsequence of length four. One way to illustrate this geometrically is with points in the plane. For the points below, the x coordinate is the position of the term in the sequence and the y coordinate is the value of the term. The points form a 3×3 grid.



Example. At a party attended by $n \geq 2$ people, some pairs of people shake hands. Show that two people shook the same number of hands.

The number of hands a person can shake is 0, 1, ... , $n-1$. At first, this would seem to be no problem, as we could assign each person one of these numbers. However, if someone shakes 0 hands, nobody can shake $n-1$ hands. Thus at most $n-1$ of the numbers 0, 1, ... , $n-1$ can be numbers of hands shaken, so by the Pigeonhole Principle, at least two people shook the same number of hands.

There is a more general version of the Pigeonhole Principle.

Theorem 4. (**The Pigeonhole Principle**) *If there are n boxes so that box i has capacity n_i, and more than $\sum n_i$ objects are distributed into the boxes, then some box contains more objects than its capacity.*

In particular, if more than nk objects that are distributed into n boxes, then some box contains at least $k+1$ objects.

The proof is essentially the same as before. The latter statement is a discrete version of the following statement about averages of real numbers.

In a set of k numbers n_i, $1 \leq i \leq k$, some number is at least $\frac{\sum n_i}{k}$ and some number is at most $\frac{\sum n_i}{k}$.

Example. Show that every year contains at least four and at most five months with five Saturdays.

A month has 28-31 days and a week has 7 days. Since $\left\lceil \frac{31}{7} \right\rceil = 5$ and $\left\lfloor \frac{28}{7} \right\rfloor = 4$, each month has four or five Saturdays. A years has 365 or 366 days. Since $\left\lceil \frac{366}{7} \right\rceil = 53$ and $\left\lfloor \frac{365}{7} \right\rfloor = 52$, each year has 52 or 53 Saturdays. If x is the number of months with five Saturdays, then $5x + 4(12 - x) = 52$ or $5x + 4(12 - x) = 53$, so x is 4 or 5.

Exercises.

1. What is the minimum number of students in a class that will guarantee that two of them have last names starting with the same letter?

2. What is the minimum number of points on a cube that will guarantee that some face contains at least two of them?

3. Assume that no man has more than a million hairs on his head. Show that two men in Houston, Texas have the same number of hairs on their heads.

4. A class contains students born in March, April, and May. How many students must there be to guarantee that two of them have the same birthday?

5. Show that if five points are inside a unit square, some pair of them have distance at most .71.

6. Show that if ten points are inside a unit square, some pair of them have distance at most .48.

7. Position n points inside a unit square so that the minimum distance between any pair of points is as large as possible when

   a. $n = 4$

   b. $n = 5$

   c. $n = 9$

   d. $n = 13$

8. Show that if five points are inside an equilateral triangle with side length 1, some pair of them have distance at most $\frac{1}{2}$.

9. Show that for any seven points on a sphere, there is a closed hemisphere that contains at least five of them.

10. Show that for any $2n+1$ points on a sphere, there is a closed hemisphere that contains at least $n+2$ of them.

11. Position four points on a sphere so that no closed hemisphere contains all of them.

12. Position five points on a sphere so that no open hemisphere contains four of them.

13. Find the minimum number of points with integer coordinates in the plane so that there must be a midpoint of one of the lines joining a pair of points that has integer coordinates.

14. Show that any set of ten ordered pairs of integers has two whose coordinates are both equal mod 3.

15. (Erdos/Szekeres [1935]) Show that any set of five points in the plane in general position has a subset of four points that form the vertices of a convex quadrilateral. (Note: The **Erdos–Szekeres conjecture** is that the minimum number of points for which any general position arrangement in the plane contains a convex subset of n points is $2^{n-2}+1$.)

16. Find eight points in the plane so that no five of them form the vertices of a convex pentagon.

17. Find all positive integers a, b, c with $a \leq b \leq c$ so that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$.

18. Find all positive integers a, b, c, d with $a < b < c < d$ so that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1$.

19. Find all sets of n integers from $[2n]$ so that none divides any other when

    a. $n = 2$

    b. $n = 3$

    c. $n = 4$

    d. $n = 5$

20. Let S be a set of n integers from $[2n]$ so that none divides any other. Show that if $a > n$ is odd, $a \in S$.

21. Let S be a set of n integers from $[2n]$ so that none divides any other. Characterize the values of n such that $a \notin S$ when

    a. $a = 1$

    b. $a = 2$

    c. $a = 3$

22. + Prove Proposition 2 using induction.

23. Show that among $2^{n-1} + 1$ subsets of $[n]$, there are two with no common elements.

24. + Characterize all sets of 2^{n-1} subsets of $[n]$ such that any pair of subsets have a common element.

25. Show that any set of $n + 1$ integers from $[2n]$ contains two that differ by 1.

26. Show that any set of $n + 1$ integers from $[kn]$ contains two that differ by less than k.

27. Show that any set of $n + 1$ integers from $[2n]$ contains two relatively prime integers.

28. Find all sets of n integers from $[2n]$ such that every pair contain a common factor.

29. (Erdos/Szekeres [1935]) Show that for any sequence of more than rs distinct numbers, there is either an increasing sequence of length more than r or a decreasing subsequence of length more than s.

30. Let σ be a permutation of $\left[n^2\right]$ whose longest monotone subsequence has length k. Show that reversing the permutation or replacing i with $n^2 + 1 - i$ for all i in the permutation results in another permutation of $\left[n^2\right]$ whose longest monotone subsequence has length k.

31. Find all permutations of $[4]$ that have no monotone subsequence of length 3.

32. Find a permutation of $\left[n^2 + 1\right]$ with exactly one monotone subsequence of length $n + 1$.

33. A baseball team scores 20 runs in the World Series. Show that they scored the same number of runs in two different games.

34. Show that a strictly increasing sequence of 100 positive integers whose largest term is at most 150 has two terms whose difference is exactly 40.

35. Show that some number in the sequence 1, 11, 111, 1111, ... is divisible by 17.

36. Show that some number in the sequence 5, 55, 555, 5555, ... is divisible by 101.

37. Given a sequence of n integers, show that there is a consecutive subsequence whose sum is divisible by n.

38. Show that any rational number has a repeating decimal expansion.

39. + (**Chinese Remainder Theorem**) Let m and n be relatively prime positive integers, and a and b be integers with $0 \le a < m$ and $0 \le b < n$. Show that there is an integer x with $x \equiv a \mod m$ and $x \equiv b \mod n$.

40. + Given a real number x, show that there are integers a, b, $1 \le a \le n$, so that $|ax - b| < \frac{1}{n}$.

41. + Show that for any $n \ge 2$, it is possible for $n - 2$ people at a party to shake different numbers of hands, and two people shake the same number of hands.

42. At a party attended by $n \ge 2$ people, everyone shakes an even number of hands. Show that three people shook the same number of hands.

43. Show that among any six people, there are three who all know each other, or three who all don't know each other.

44. Show that there is a group of five people such that there is no group of three who all know each other, and no group of three who all don't know each other.

45. At a small college, the average student has eight friends. Show that there is a group of students such that each of them has at least five friends in the group.

46. Show that your father and mother have a common ancestor within the last 30 generations.

47. Two disks are each partitioned into 50 sectors of the same size. Disk 1 has 25 sectors colored red and 25 colored white. The 50 sectors of disk 2 are arbitrarily colored red and white. Prove that the smaller disk can be rotated so that at least 25 sectors of the two disks match in color.

48. + Suppose that all points in the plane with integer coordinates are colored with three colors. Show that there is a rectangle with all vertices colored the same.

References

[1935] Erdos, Paul; Szekeres, George (1935), "A combinatorial problem in geometry", Compositio Mathematica, 2: 463–470.

[1959] A. Seidenberg, "A simple proof of a theorem of Erdos and Szekeres" Journal of the London Mathematical Society, 34 (1959) 352.