1 Uniquely Colorable Graphs

A complete k-partite graph has only one partition into k color classes. Thus it has only one way to be colored, provided that we don't care which particular colors are used.

Definition 1. A graph is **uniquely k-colorable** if any k-coloring produces the same vertex partition. A graph is **uniquely colorable** if any minimum coloring produces the same vertex partition.

Adding edges consistent with a minimum coloring of a graph limits the possible minimum colorings, until eventually the graph is uniquely colorable. Thus uniquely k-colorable graphs are a larger class containing maximal k-chromatic graphs. A sufficiently large size guarantees a graph is uniquely k-colorable.

Theorem 2. (Bollobas [1978]) *If G is a k-colorable graph with $\delta(G) > \frac{3k-5}{3k-2}n$, then G is uniquely k-colorable.*

Proof. We use induction on k. Let $k = 2$. If G is not connected, let H be a component of G of order $m \le \frac{n}{2}$. Then $\delta(H) > \frac{m}{2}$, so H contains a triangle. But this is impossible, so G is connected and uniquely 2-colorable.

Now let $k \ge 3$ and suppose the result holds for smaller values of k. Given $x \in V(G)$, let $G_x = G[N(x)]$. Denote the order of G_x by n_x. Then $n_x > \frac{3k-5}{3k-2}n$, and

$$d_{G_x}(y) \ge \frac{3k-5}{3k-2}n - (n - n_x) = n_x - \frac{3}{3k-2}n > \frac{3(k-1)-5}{3(k-1)-2}n_x.$$

Therefore by the induction hypothesis G_x is uniquely $k-1$-colorable.

Now let u_1 and u_2 be vertices of G. As $d(u_i) > \frac{3k-5}{3k-2}n \ge \frac{4}{7}n > \frac{1}{2}n$, there is a vertex x adjacent to bothu_1 and u_2, so u_1 and u_2 belong to G_x. Now a k-coloring of G always gives a $k-1$-coloring of G_x. As this $k-1$-coloring is unique, either u_1 and u_2 get the same color or they get different colors, independently of the k-coloring of G. Thus G is uniquely colorable. □

One way to produce a uniquely colorable graph is to have many overlapping cliques. Trees are uniquely colorable, as every edge corresponds to a 2-clique. If G is uniquely k-colorable, and a vertex v of degree $k-1$ is added so that it is adjacent to vertices in all but one color class, the new graph is uniquely k-colorable. Thus k-trees are uniquely colorable. Adding a vertex with degree less than $k-1$ yields more than one possible color for this vertex, so a uniquely k-colorable graph G has $\delta(G) \ge k-1$.

Lemma 3. (HHR [1969]) *In a uniquely colorable graph, any two color classes induce a connected graph.*

Proof. If not, the colors could be exchanged on one component of the graph they induce. □

Bollobas [1978] showed that any graph with $\delta(G) > \frac{k-2}{k-1}n$ so that each pair of color classes induce a connected subgraph is uniquely k-colorable.

Theorem 4. (Shaoji [1990]) *A uniquely $k+1$-colorable graph has $m \ge kn - \frac{k(k+1)}{2}$.*

Proof. Let G be uniquely $k+1$-colorable with color classes V_i. Each edge of G is in exactly one subgraph induced by two color classes. Thus

$$\begin{aligned} m(G) &= \sum_{i \ne j} m(G[V_i \cup V_j]) \\ &\ge \sum_{i \ne j} (|V_i \cup V_j| - 1) \\ &= \sum_{i \ne j} |V_i \cup V_j| - \binom{k+1}{2} \\ &= kn - \frac{k(k+1)}{2}. \end{aligned}$$

□

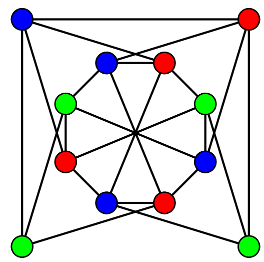
Uniquely colorable maximal k-degenerate graphs have $m = kn - \frac{k(k+1)}{2}$. Note that any uniquely 4-colorable graph has $m \ge 3n - 6$. Thus any uniquely 4-colorable planar graph is maximal planar (Chartrand/Geller [1969]). A natural example of such graphs are planar 3-trees.

Theorem 5. (Fowler [1998]) *Every uniquely 4-colorable planar graph is a 3-tree.*

The proof of this theorem uses techniques from the proof of the Four Color Theorem. In fact, Fowler showed that any maximal planar graph that is not a 3-tree has at least two 4-colorings, which generalizes the Four Color Theorem.

Any 3-colorable maximal planar graph is uniquely 3-colorable, but maximality is not required. Aksionov [1977] showed that any uniquely 3-colorable planar graph with $n \geq 5$ has at least three triangles, and all such graphs are maximal 2-degenerate. LZSX [2017] proved that any uniquely 3-colorable planar graph with at most four triangles has two adjacent triangles.

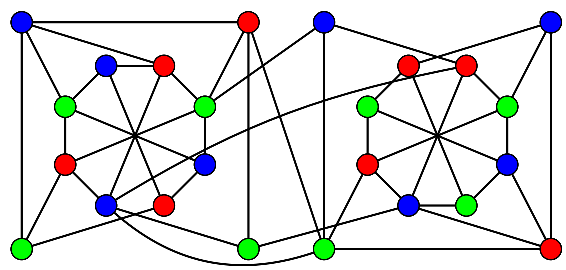
Surprisingly, there are also triangle-free uniquely 3-colorable graphs. The example below has one edge deleted from the Chvatal graph (HHR [1969]).



This result has been generalized to large girths using a probabilistic argument.

Theorem 6. (Nesetril [1973], Erdos [1974], Bollobas/Sauer [1976]) *For all $k \geq 2$ and $g \geq 3$ there is a uniquely k-colorable graph with girth at least g.*

The graph above has $m > 2n - 3$. AMS [2001] showed that the following graph is uniquely 3-colorable, triangle-free, and has $m = 2n - 3$.



Li/Xu [2016] found an example of such a graph with order 16.

Exercises

1. Characterize uniquely 2-colorable graphs.
2. Determine which graphs in the following classes are uniquely colorable.
   a. K_n
   b. C_n
   c. W_n
3. Show that if G is k-critical and uniquely k-colorable, then $G = K_k$.
4. Prove or disprove: If G is $(k+1)$-critical, then $G - e$ is uniquely k-colorable for any edge e.
5. (Bollobas [1978]) Show that the bound in Theorem 2 is sharp for all k.
6. (Bollobas [1978]) Show that the graph H formed from $K_3 \square K_2$ by substituting $\overline{K}_l$ for each vertex. Form G by joining H to $K_{3l,\ldots,3l}$ (there are $k-3$ partite sets). Show that this graph is not uniquely k-colorable and $\delta(G) = \frac{k-2}{k-1}n$. (Note: Bollobas showed that any graph with larger minimum degree so that each pair of color classes induce a connected subgraph is uniquely k-colorable.)
7. Prove or disprove: A maximal k-degenerate graph is uniquely colorable if and only if it is a k-tree.

8. Prove or disprove: A chordal graph is uniquely colorable if and only if it is a k-tree.

9. Let G be a uniquely k-colorable graph with $d(v) = k-1$ for some vertex v. Show that $G - v$ is uniquely colorable.

10. Show that a uniquely k-colorable graph G has $\kappa(G) \geq k-1$.

11. Show that G and H are uniquely colorable graphs if and only if $G+H$ is uniquely colorable.

12. (Li/Xu [2016]) Let $G_1$and G_2 be uniquely 3-colorable graphs and u_i and v_i be differently colored vertices in the unique coloring of G_i, $i \in \{1,2\}$.

    a. Form G by identifying u_1 with u_2 and v_1 with v_2. Show that G is uniquely 3-colorable.

    b. Form G by identifying u_1 with u_2 and adding edge v_1v_2. Show that G is uniquely 3-colorable.

    c. Describe a method for constructing a uniquely 3-colorable graph from $G_1 \cup G_2$ by adding three edges.

13. (Chartrand/Geller [1969]) Show that an outerplanar graph G of order $n \geq 3$ is uniquely 3-colorable if and only if G is maximal outerplanar.

14. (Chartrand/Geller [1969]) Show that if a 2-connected 3-chromatic plane graph G and at most one region of G is not a triangle, then G is uniquely 3-colorable. Show that this statement is false if there are two non-triangular regions.

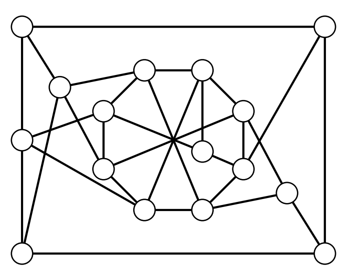
15. Verify that the graph formed by deleting one edge from the outer 4-cycle of the Chvatal graph is uniquely 3-colorable.

16. (Chao/Chen [1993]) Show that for all $n \geq 12$, there is a uniquely 3-colorable triangle-free graph.

17. (Chao/Chen [1993]) Show that there is a 5-regular uniquely 3-colorable triangle-free graph of order 24.

18. Prove or disprove: There is a cubic uniquely 3-colorable graph.

19. + (Li/Xu [2016]) Show that the following graph is uniquely 3-colorable, triangle-free, and has $m = 2n-3$. (Note: Li/Xu use a construction based on this graph to show that there are infinitely many 4-regular uniquely 3-colorable triangle-free graphs.)



20. + Verify that the graph of AMS [2001] is uniquely 3-colorable.

References

[2001] S. Akbari, V. S. Mirrokni, B. S. Sadjad, Kr-Free uniquely vertex colorable graphs with minimum possible edges. J. Combin. Theory Ser. B 82 (2001), 316–318.

[1977] V. A. Aksionov, On uniquely 3-colorable planar graphs, Discrete Math 20 (1977) 209-216.

[1978] B. Bollobas, Uniquely colorable graphs. J. Combin. Theory Ser. B 25 (1978), 54-61.

[1976] B. Bollobas, N. W. Sauer, Uniquely colourable graphs with large girth, Canad. J. Math., 28 (1976), 1340-1344.

[1993] C.-Y. Chao and Z. Chen, On Uniquely 3-Colorable Graphs. Disc. Math. 112 (1993), 21-27.

[1969] G. Chartrand and D. Geller, On uniquely colorable planar graphs. J. Combin. Theory 6 3 (1969), 271-278

[1974] P. Erdos, Some new applications of probability methods to combinatorial analysis and graph theory, Proc. 5th southeast. Conf. Comb., Graph Theor., Comput.; Boca Raton, 10 (1974), 39–51.

[1998] T. Fowler, Unique Coloring of Planar Graphs, Ph.D. thesis, Georgia Institute of Technology Mathematics Department (1998).

[1969] F. Harary, S. Hedetniemi, and R. Robinson, Uniquely Colorable Graphs. J. Combin. Th. 6 (1969), 264-270.

[2016] Z. Li and J. Xu, Constructions of uniquely 3-colorable graphs, 2016 IEEE First International Conference on Data Science in Cyberspace, 142-147.

[2017] Z. Li, E. Zhu, Z. Shao, and J. Xu, A note on uniquely 3-colorable planar graphs. Int. J. Comput. Math. 94 (2017), 1028–1035.

[1973] J. Nesetril, On uniquely colorable graphs without short cycles, Casopis. Pest. Math., 98 (1973), 122–125.

[1990] Xu Shaoji, The size of uniquely colorable graphs. J. Combin. Theory Ser. B 50 2 (1990), 319-320.